

# Torsion in Structural Design

## 1. Introduction

### 1.1. Problems in Torsion

The role of torsion in structural design is subtle, and complex. Some torsional phenomena include

- (a) Twist of beams under loads not passing through the shear center
  - (b) Torsion of shafts
  - (c) Torsional buckling of columns
  - (d) Lateral torsional buckling of beams
- Two main types of situation involve consideration of torsion in design
    - (1) Member's main function is the transmission of a primary torque, or a primary torque combined with bending or axial load (Cases (a) and (b) above.)
    - (2) Members in which torsion is a secondary undesirable side effect tending to cause excessive deformation or premature failure. (Cases (c) and (d) above.)

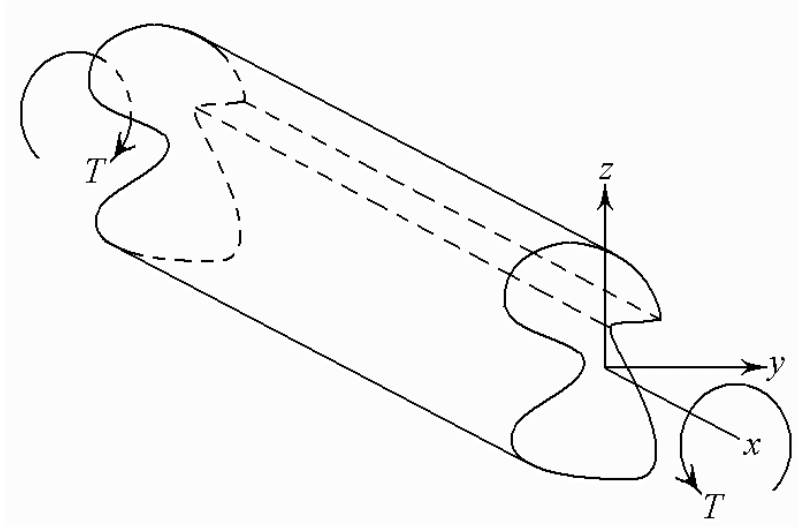
## 1.2. Development of Torsional analysis- A few key contributors

- 1853 - French engineer Adhemar Jean Barre de Saint-Venant presented the classical torsion theory to the French Academy of Science
- 1899 - A. Michell and L. Prandtl presented results on flexural-torsional buckling
- 1905 - S. P. Timoshenko presented a paper on the effects of warping torsion in I beams
- 1909 - C. Bach noted the existence of warping stresses not predicted by classical torsion theory when the shear center and centroid do not coincide.
- 1929 - H. Wagner began to develop a general theory of flexural torsional buckling
- V. Z. Vlasov (1906-1958) developed the theory of general bending and twisting of thin walled beams
- 1944 - von Karman and Christensen developed a theory for closed sections (approximate theory)
- 1954 - Benscoter developed a more accurate theory for closed sections.

Numerous other contributors, these are just a few highlights.

## 2. Uniform Torsion of Prismatic Sections

Consider a prismatic shaft under constant twisting moment along its length.



Classical theory due to St Venant. Assume

- Cross-sections do not distort in plane during twisting, so every point in the section rotates (in plane) through angle  $\phi(x)$  about the center of twist.
- Out of plane warping is not constrained
- Out of plane warping does not vary along the bar

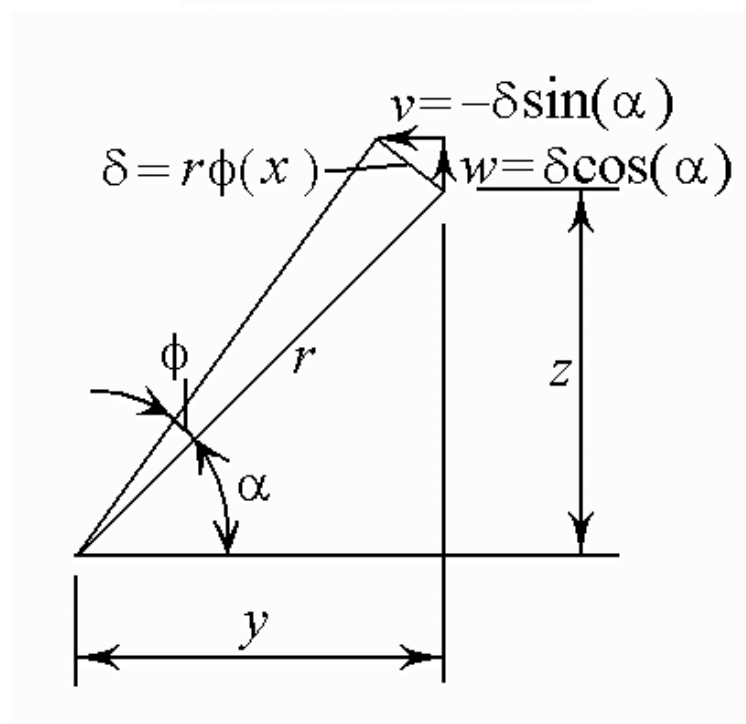
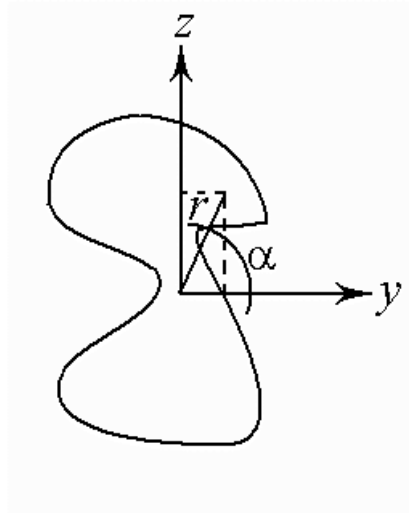
The resulting displacement field is

$$u = \frac{d\phi}{dx} \omega(y, z)$$

$$v = -z\phi(x) = -zx \frac{d\phi}{dx}$$

$$w = y\phi(x) = yx \frac{d\phi}{dx}$$

In plane displacements  $v$  and  $w$  are seen from the figure below...



Out of plane distortion (warping) of the section is assumed to vary with the *rate* of twist

$$\theta = \frac{d\phi(x)}{dx} = \text{constant} \rightarrow \phi(x) = x\theta$$

and to be a function of the position  $(y,z)$  on the cross-section only. Several models may be constructed

- Warping function model.
- Conjugate Harmonic function model
- St Venant's stress function model

## Warping Function Model

Substituting the displ. fields into the diff. eq. of equilibrium from elasticity, we obtain

$$\frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2} = \nabla^2 \omega = 0 \quad (\text{Laplace's equation})$$

with b.c.

$$\frac{\partial \omega}{\partial n} = za_{ny} - ya_{nz}$$

where  $n$  is the normal direction to the boundary, and  $(a_{ny}, a_{nz})$  are the components of the unit normal vector  $\mathbf{n}$  on the boundary.

It can be shown that the St Venant torsional stiffness of the section is given by

$$J = \iint_A y^2 + z^2 + y \frac{\partial \omega}{\partial y} - z \frac{\partial \omega}{\partial z} dA$$

and that the angle of twist is related to the torque by

$$T = \frac{JG\phi}{L} \quad \text{or} \quad T = JG \frac{d\phi}{dx}$$

a result that reduces to the usual polar moment of inertia when the section becomes circular, and the warping function vanishes.

Problem: The eqn. is hard to solve with the b.c given.

## St Venant's Stress Function Model

Assume that the non-zero stresses  $\tau_{xy}, \tau_{xz}$  are related to a stress function  $\Psi(y, z)$  by

$$\tau_{xy} = \frac{\partial \Psi}{\partial z} \quad \tau_{xz} = -\frac{\partial \Psi}{\partial y}$$

The function  $\Psi(y, z)$  automatically satisfies equilibrium. In order for the resulting displacements to be compatible (i.e. satisfy continuity) the d.e.

$$\frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} = -2G \frac{d\phi}{dx} = -2G\theta$$

be satisfied, where  $G$ =the shear modulus.  
The boundary conditions for this model are

$$\Psi(x, y) = \text{constant on } \Gamma$$

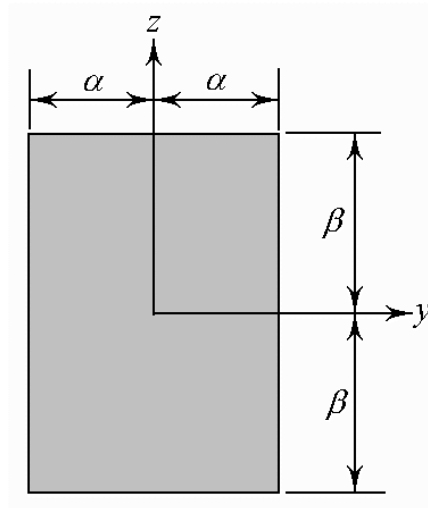
where  $\Gamma$  is the boundary of the section. In many cases, it is convenient to simply take  $\Psi=0$  on the boundary.

Given the stress function, it can be shown that

$$J = \iint_A \Psi(y, z) dy dz$$

over the section.

This d.e. is somewhat easier to solve *because of the simpler b.c.* We are particularly interested in rectangular sections. Consider such a section, of dimension  $2\alpha \times 2\beta$ , as shown.



It can be shown (using a Levy type solution) that, for this section

$$\Psi = \frac{32G\phi\alpha^2}{\pi^3} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} (-1)^{(n-1)/2} \left[ 1 - \frac{\cosh(n\pi z / 2\alpha)}{\cosh(n\pi\beta / 2\alpha)} \right] \cos\left(\frac{n\pi y}{2\alpha}\right)$$

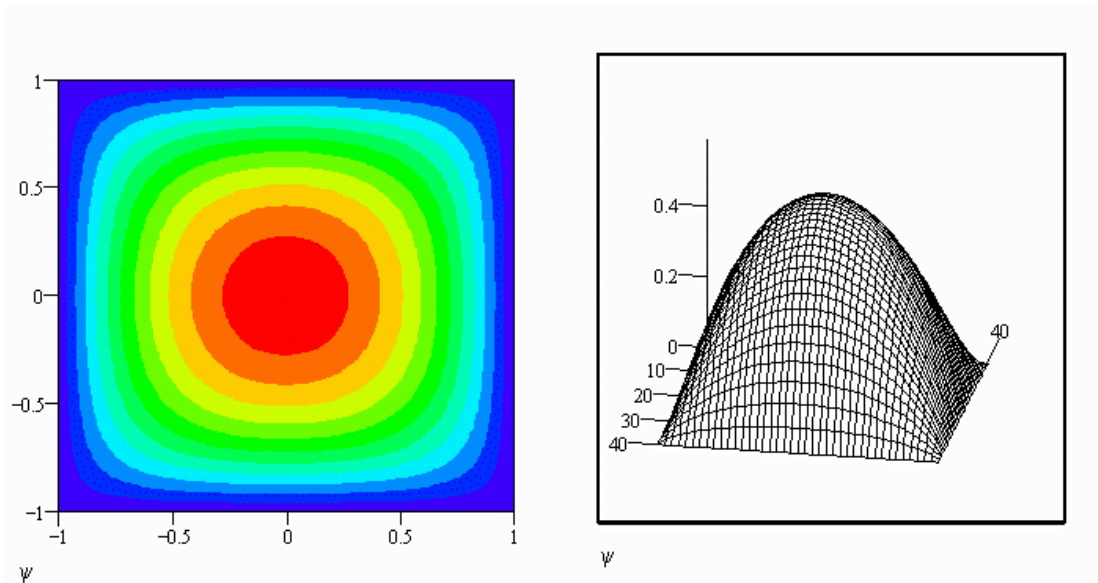
The corresponding stresses are

$$\tau_{xz} = \frac{16G\phi\alpha^2}{\alpha\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} (-1)^{(n-1)/2} \left[ 1 - \frac{\cosh(n\pi z / 2\alpha)}{\cosh(n\pi\beta / 2\alpha)} \right] \sin\left(\frac{n\pi y}{2\alpha}\right)$$

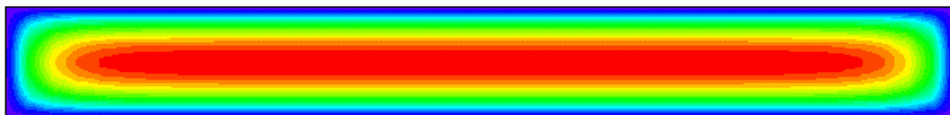
$$\tau_{xy} = \frac{16G\phi\alpha^2}{\alpha\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} (-1)^{(n-1)/2} \left[ -\frac{\sinh(n\pi z / 2\alpha)}{\cosh(n\pi\beta / 2\alpha)} \right] \cos\left(\frac{n\pi y}{2\alpha}\right)$$



$\Psi$  is shown below for two sections



Stress function for a square section



Stress function for a section with aspect ratio 10

Assume overall dimensions  $b=2\alpha$ ,  $t=2\beta$ . Assume that  $b \geq t$ .  
In general, it is useful to write

$$\tau_{\max} = ktG \frac{d\phi}{dx}$$

$$J = k_1 t^3 b$$

Then, since  $\tau_{\max} \propto T$ , we may write

$$\tau_{\max} = \frac{T}{k_2 t^2 b}$$

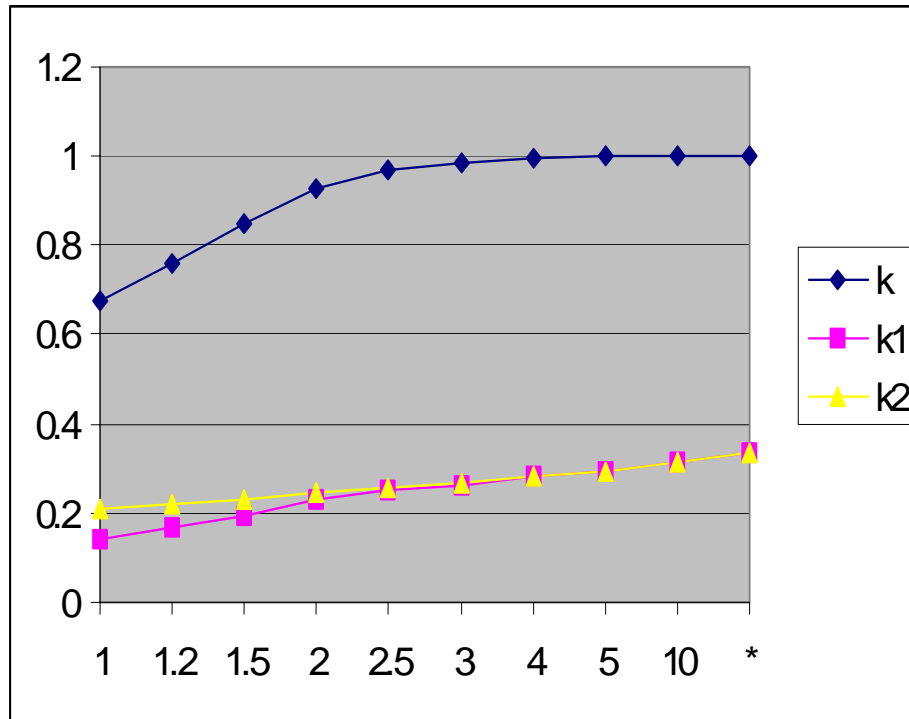
Values of the coefficients are tabulated below for different aspect ratios

**Table 1 Coefficients for Torsion of a Rectangular Section**

$b/t$	$k$	$k_1$	$k_2$
1.0	0.675	0.1406	0.208
1.2	0.759	0.166	0.219
1.5	0.848	0.196	0.231
2.0	0.930	0.229	0.246
2.5	0.968	0.249	0.258
3	0.985	0.263	0.267
4	0.997	0.281	0.282
5	0.999	0.291	0.291
10	1.000	0.312	0.312
$\infty$	1.000	0.333	0.333

It is seen that for the thin section, the response away from the ends is almost independent of  $y$ . Hence, a simplified model takes the form

$$\frac{d^2\Psi}{dz^2} = -2G\theta \quad \text{subject to} \quad \Psi(\pm\beta) = 0$$



This leads to the simple solution

$$\Psi(z) = G\theta\beta^2 \left[ 1 - \frac{z^2}{\beta^2} \right] \text{ with resulting shear stress}$$

$$\tau_{xy} = -2G\theta\beta(z/\beta)$$

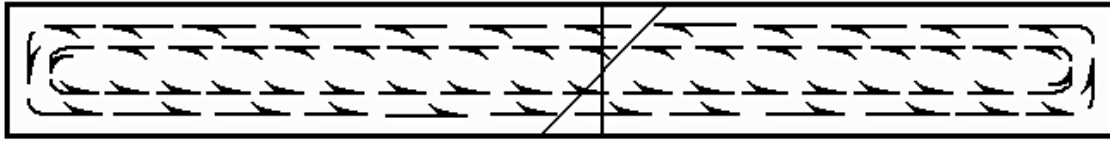
Hence, at  $z=t/2$ ,

$$\tau_{\max} = Gt\theta = Gt \frac{d\phi}{dx}$$

Integrating this approximate  $\Psi$  function over the area,

$$J = \frac{1}{3}bt^3$$

consistent with the limit of the 2-D solution as the width to thickness ratio approaches  $\infty$ . The shear stresses in a narrow rectangular beam are as shown below



### Thin-walled open section beams

Some thin walled open sections: angles, channels, W and S sections, T sections, etc.

- Key result 1. The section torsional stiffness  $J$  is approximately equal to the sum of the  $J$ s for the constituent thin walled plates. Thus,

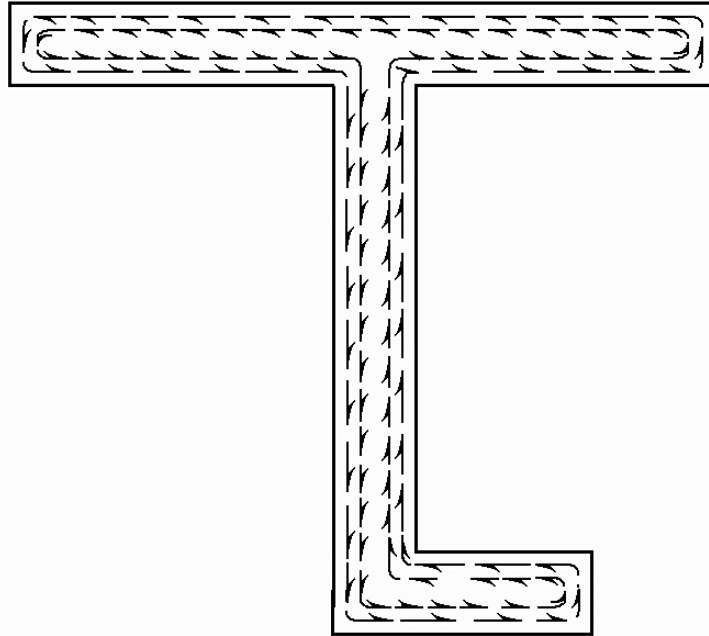
$$J = \frac{1}{3} \sum_{i=1}^n b_i t_i^3$$

- Key result 2. The maximum shear stress is estimated as

$$\tau_{\max} = \frac{T t_{\max_i}}{J}$$

except that larger stresses may occur at the corners .

- Key result 3. The shear flow around the section, caused by St Venant torsion, is as shown below.



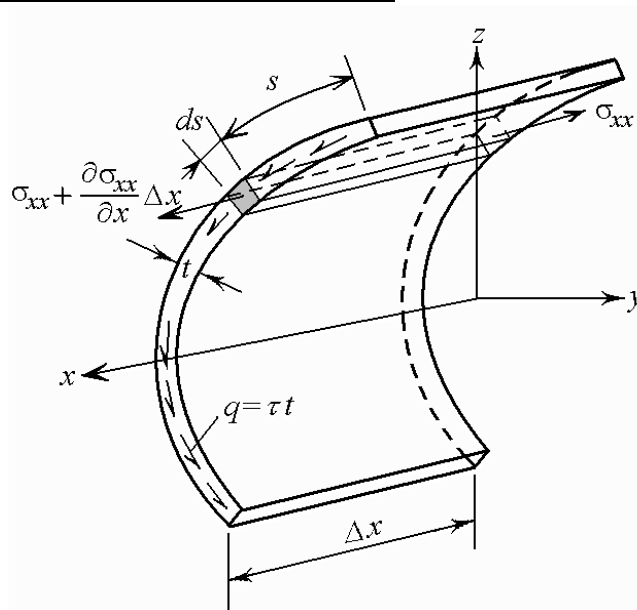
That is, the stresses vary linearly through the thickness on any of the constituent plates, achieving a maximum on each edge.

### 3. Shear Stresses due to bending - thin walled open sections

- Provides important background needed for analysis of torsion
- Three items are needed...
  - shear stresses and shear flow concept
  - shear center
  - sectorial moments

These items can be obtained from a generalization of the analysis studied in undergraduate Mechanics of Materials. We summarize the results of the analysis below.

#### Normal Stresses and Resultants



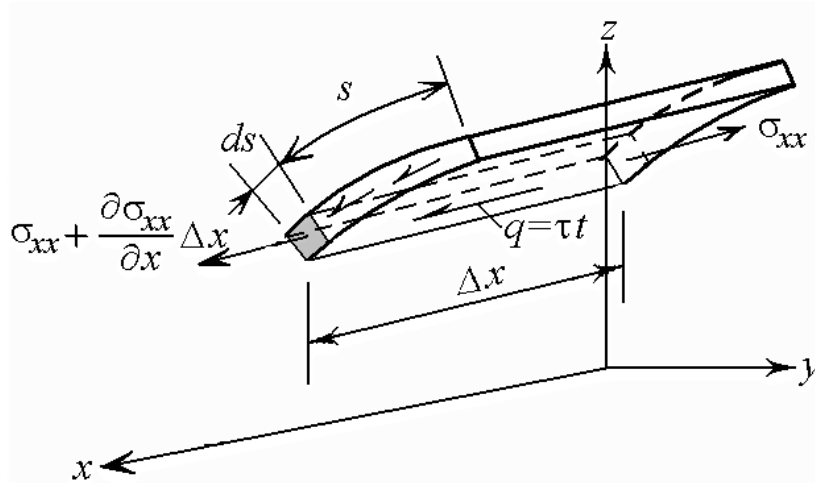
From generalized analysis of beams,

$$\sigma_{xx} = -\frac{I_{yy}M_z - I_{yz}M_y}{I_{yy}I_{zz} - I_{yz}^2}y - \frac{I_{zz}M_y - I_{yz}M_z}{I_{yy}I_{zz} - I_{yz}^2}z$$

where

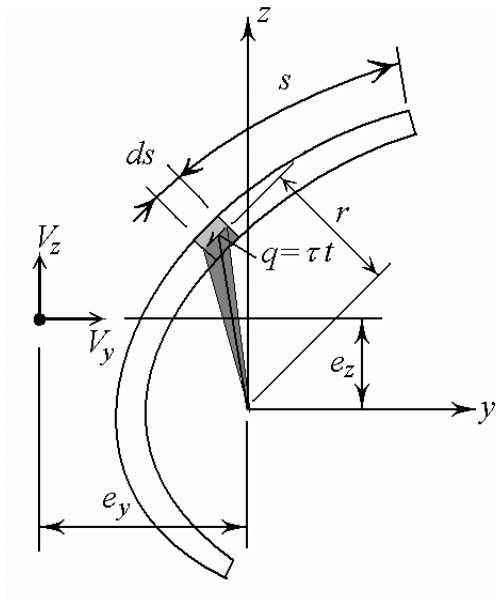
$$I_{yy} = \int_A z^2 dA \quad I_{zz} = \int_A y^2 dA = \text{the moments of inertia.}$$

$$I_{yz} = \int_A yz dA = \text{product of inertia}$$



From equilibrium of the slice of the section cut above,

$$q = \tau t = \frac{I_{yy} Q_z(s) - I_{yz} Q_y(s)}{I_{yy} I_{zz} - I_{yz}^2} V_y + \frac{I_{zz} Q_y(s) - I_{yz} Q_z(s)}{I_{yy} I_{zz} - I_{yz}^2} V_z$$



where

$$Q_y = \int_{A^*(s)} z dA = \int_0^s z t d\bar{s}, \quad Q_z = \int_{A^*(s)} y dA = \int_0^s y t d\bar{s}$$

$q$  is the *shear flow* on the section. If the shear resultants act through the shear center, there is no twist.

Summing moments about the centroid, it can be shown that the coordinates of the shear center are

$$e_y = -2 \frac{I_{zz} I_{\omega y} - I_{yz} I_{\omega z}}{I_{yy} I_{zz} - I_{yz}^2}$$

$$e_z = 2 \frac{I_{yy} I_{\omega z} - I_{yz} I_{\omega y}}{I_{yy} I_{zz} - I_{yz}^2}$$

In the above,

$$I_{\omega y} = \frac{1}{2} \int_0^S Q_y(s) r ds = \int_0^S Q_y(s) d\omega(s)$$

$$I_{\omega z} = \frac{1}{2} \int_0^S Q_z(s) r ds = \int_0^S Q_z(s) d\omega(s)$$

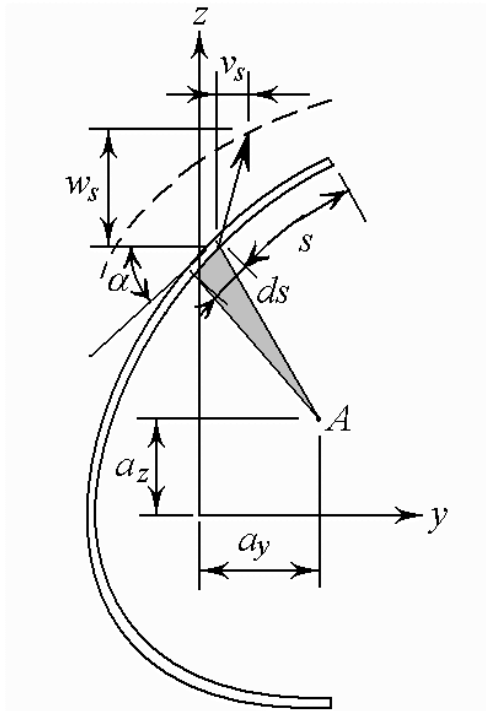
are the *sectorial products of inertia* of the section, and  $r(s) ds = 2 d\omega(s)$  define the *sectorial area* (the area shaded in the figure above.)

$r(s)$  is the *perpendicular distance* from the origin to the force  $q ds$  needed in calculating moments.



## 4. Bending and torsion of open sections

For combined bending and torsion, a number of additional sectorial area properties are needed for the section.



Let point A undergo displacements  $v$ ,  $w$ , and rotation  $\phi$ . A different point  $s$ , located at coordinates  $(y, z)$  displaces by

$$v_s = v - (z - a_z)\phi$$

$$w_s = w + (y - a_y)\phi$$

This can be resolved into a normal, and a tangential displacement component. The tangential component of displacement is

$$\eta = v \frac{dy}{ds} + w \frac{dz}{ds} + r\phi$$

The shear strain in the tangential plane is given by

$$\gamma_{xs} = \frac{\partial u}{\partial s} + \frac{\partial \eta}{\partial x} \approx 0 \quad (\text{Wagner's assumption})$$

From which the out of plane displacement is approximately

$$u = -y(s) \frac{dv}{dx} - z(s) \frac{dw}{dx} - 2\omega(s) \frac{d\phi}{dx} + u_0(x)$$

In the above eqn.

$$2\omega(s) = \int_0^s r d\bar{s} = 2 \int_0^s d\omega(\bar{s})$$

is the sectorial area at point  $s$ . The resulting normal stress is then

$$\sigma_{xx} = E \frac{\partial u}{\partial x} = E \left( -y \frac{d^2 v}{dx^2} - z \frac{d^2 w}{dx^2} - 2\omega \frac{d^2 \phi}{dx^2} + \frac{du_0}{dx} \right)$$

Integrating the normal stress over the area leads to the axial and bending resultants.

These may be written as

$$N_x = EA \frac{du_0}{dx} - ES_\omega \frac{d^2 \phi}{dx^2}$$

$$M_y = -EI_{yz} \frac{d^2 v}{dx^2} - EI_{yy} \frac{d^2 w}{dx^2} + 2EI_{\omega y} \frac{d^2 \phi}{dx^2}$$

$$M_z = -EI_{zz} \frac{d^2 v}{dx^2} - EI_{yz} \frac{d^2 w}{dx^2} + 2EI_{\omega z} \frac{d^2 \phi}{dx^2}$$

In the above, the first sectorial moment

$$S_\omega = \iint_A 2\omega(s) dA$$

has been used. The sectorial products of inertia  $I_{\omega y}, I_{\omega z}$  were defined in connection with the shear center.

Solving for the displacements,

$$\frac{du_0}{dx} = \frac{N_x}{AE} + \frac{S_\omega}{A} \frac{d^2\phi}{dx^2}$$

$$\frac{d^2v}{dx^2} = -\frac{M_z I_{yy} - M_y I_{yz}}{E(I_{yy} I_{zz} - I_{yz}^2)} + 2 \frac{I_{\omega z} I_{yy} - I_{\omega y} I_{yz}}{I_{yy} I_{zz} - I_{yz}^2} \frac{d^2\phi}{dx^2}$$

$$\frac{d^2w}{dx^2} = -\frac{M_y I_{zz} - M_z I_{yz}}{E(I_{yy} I_{zz} - I_{yz}^2)} + 2 \frac{I_{\omega y} I_{zz} - I_{\omega z} I_{yz}}{I_{yy} I_{zz} - I_{yz}^2} \frac{d^2\phi}{dx^2}$$

The coefficients multiplying  $\phi''$  are the distances from  $A$  to the shear center, so if point  $A$  is taken at the shear center, these terms disappear, and the equations for  $v$  and  $w$  reduce to

$$\frac{d^2v}{dx^2} = -\frac{M_z I_{yy} - M_y I_{yz}}{E(I_{yy} I_{zz} - I_{yz}^2)}$$

$$\frac{d^2w}{dx^2} = -\frac{M_y I_{zz} - M_z I_{yz}}{E(I_{yy} I_{zz} - I_{yz}^2)}$$

The equation for  $u_0''$  can be written in alternative form as

$$\frac{du_0}{dx} = \frac{N_x}{AE} + 2\omega_0 \frac{d^2\phi}{dx^2}$$

where  $\omega_0$  = the sectorial area between the outside edge of the section and the sectorial centroid, a point for which  $S_\omega$  vanishes. If the  $s$  origin were taken at a sectorial centroid, instead of at an outside edge, then this term would disappear.

## Shear Stresses and Resultants

For a section under bending and twisting, the total shear stress at any point is

$$\tau_{xs} = (\tau_{xs})_{SV} + \frac{q}{t}$$

where

$(\tau_{xs})_{SV}$  = St Venant torsion stresses

$\frac{q}{t}$  = shear stresses caused by shear flow on the section.

Integrating the twisting moments caused by the shearing stresses,

$$T = GJ \frac{d\phi}{dx} + T_{\omega}$$

where  $T_{\omega}$  is the torque caused by constrained warping induced shear flow. It is sometimes convenient to express this in terms of applied torque per unit length  $t(x)$ . Here,

$$t(x) = -\frac{dT}{dx} = -GJ \frac{d^2\phi}{dx^2} - \frac{dT_{\omega}}{dx}$$

If the free edge is taken as the origin of the  $s$  coordinate, the shear flow is given by

$$q(s) = -\int_0^s \frac{\partial \sigma_{xx}}{\partial x} dA$$

Hence,

$$q(s) = - \left( \frac{I_{yy}Q_z - I_{yz}Q_y}{I_{yy}I_{zz} - I_{yz}^2} V_y + \frac{I_{zz}Q_y - I_{yz}Q_z}{I_{yy}I_{zz} - I_{yz}^2} V_z \right) - \frac{Q_\omega}{C_\omega} V_\omega$$

where

$$W_\omega = \iint_A 2\sigma_{xx}(\omega_0 - \omega)dA = \text{the bimoment}$$

$$V_\omega = \frac{d}{dx} W_\omega = \frac{d}{dx} \iint_A 2\sigma_{xx}(\omega_0 - \omega)dA$$

$$Q_\omega = \int_0^s 2(\omega_0 - \omega)dA = \text{the first sectorial moment}$$

$$C_\omega = \iint_A 4(\omega_0 - \omega)^2 dA = \text{the warping constant}$$

Integrating  $q$  over the section, and recognizing that the bending related shear flow vanishes if the load passes through the shear center (which we may always impose) we obtain, after some manipulation

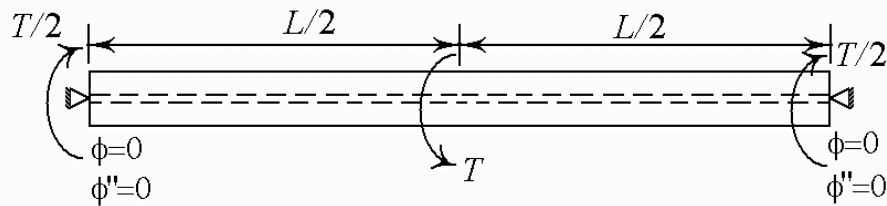
$$T_\omega = -EC_w \frac{d^3\phi}{dx^3}$$

from which the equation for twist of a thin-walled section

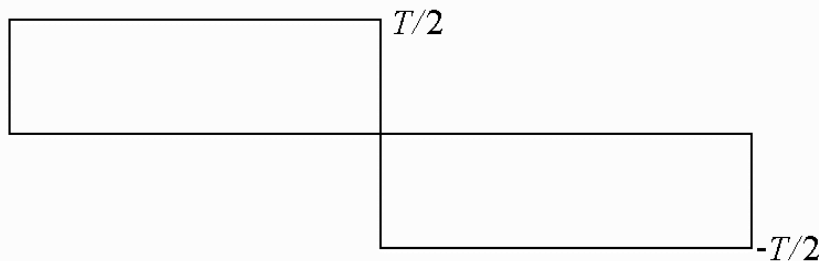
$$EC_w \frac{d^4\phi}{dx^4} - GJ \frac{d^2\phi}{dx^2} = t(x)$$

is obtained.

Example Consider an I beam, simply supported at each end, free to warp at the ends, loaded by a twisting moment in the middle.



(a) I Beam under concentrated torque at midspan

(b) Torsional moment as a function of  $x$ 

The conditions at the left and right end are similar to those at the end of a simply supported beam in bending. Hence

$$\begin{aligned} \phi(0) &= 0 & \phi(L) &= 0 \\ \phi''(0) &= 0 & \phi''(L) &= 0 \end{aligned}$$

Rather than use the fourth order differential equation which has discontinuous load at midspan, use the third order differential equations

$$\begin{aligned} EC_{\omega}\phi''' - GJ\phi' &= T/2 & 0 \leq x \leq L/2 \\ EC_{\omega}\phi''' - GJ\phi' &= -T/2 & L/2 < x \leq l \end{aligned}$$

The homogeneous eqn.

$$EC_{\omega}\phi''' - GJ\phi' = 0$$

has the solution

$$\phi_h(x) = C_0 + C_1 \sinh(\lambda x) + C_2 \cosh(\lambda x)$$

where  $\lambda = \sqrt{\frac{GJ}{EC_\omega}}$ . Since the r.h.s. of the non-

homogeneous equations are constant, and the lowest order derivative of  $\phi$  in the o.d.e. is one, assume a particular solution

$$\phi_p = \alpha x$$

Substituting into the o.d.e.'s,

$$GJ\alpha = T/2 \rightarrow \alpha = T/2GJ \quad 0 \leq x \leq L/2$$

$$GJ\alpha = -T/2 \rightarrow \alpha = -T/2GJ \quad L/2 < x \leq L$$

Hence, the general solution is given by

$$\phi = C_0 + C_1 \sinh(\lambda x) + C_2 \cosh(\lambda x) + Tx/2GJ \quad 0 \leq x \leq L/2$$

$$\phi = \bar{C}_0 + \bar{C}_1 \sinh(\lambda x) + \bar{C}_2 \cosh(\lambda x) - Tx/2GJ \quad L/2 < x \leq L$$

The two solutions must be identical at midspan, and by symmetry,  $\phi'(L/2) = 0$ .

Consider the left half of the beam first. For this portion of the beam, the b.c. at the origin are

$$\phi(0) = 0 \rightarrow C_0 + C_2 = 0$$

$$\phi''(0) = 0 \rightarrow \lambda^2 C_2 = 0$$

Hence,  $C_0 = C_2 = 0$ , and the solution reduces to

$$\phi = C_1 \sinh(\lambda x) + Tx/2GJ$$

Then, the symmetry condition at midspan

$$\phi'(L/2) = 0 = \lambda C_1 \cosh(\lambda L/2) + T/2GJ$$

yields  $C_1 = -\frac{T}{2GJ\lambda} \frac{1}{\cosh(\lambda L/2)}$ .

Hence, for the left half of the beam,

$$\phi = \frac{T}{2JG\lambda} \left[ \lambda x - \frac{\sinh(\lambda x)}{\cosh(\lambda L/2)} \right]$$

For the right hand side of the beam, we may place the origin at the right end, running to the left.

Then, it can be shown that the resulting solution (after converting to the original coordinate system) is

$$\phi = \frac{T}{2JG\lambda} \left[ \lambda(L-x) - \frac{\sinh(\lambda(L-x))}{\cosh(\lambda L/2)} \right]$$

Next, consider the relative magnitudes of the two components of torsion over the span. We have, for the left half of the span,

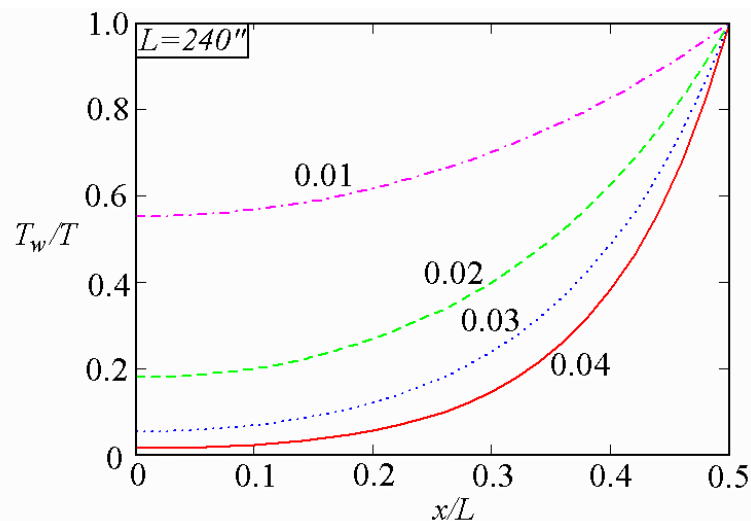
$$T_{sv} = GJ\phi' = \frac{T}{2} \left[ 1 - \frac{\cosh(\lambda x)}{\cosh(\lambda L/2)} \right]$$



$$T_w = -EC_\omega \phi''' = \frac{T}{2} \frac{\cosh(\lambda x)}{\cosh(\lambda L/2)}$$

For a typical W section, the  $\lambda$  values range from roughly .04 for a heavy column section to less than .01 for some very slender beams.

Contributions of the warping torsion are shown below for a 20' long beam.



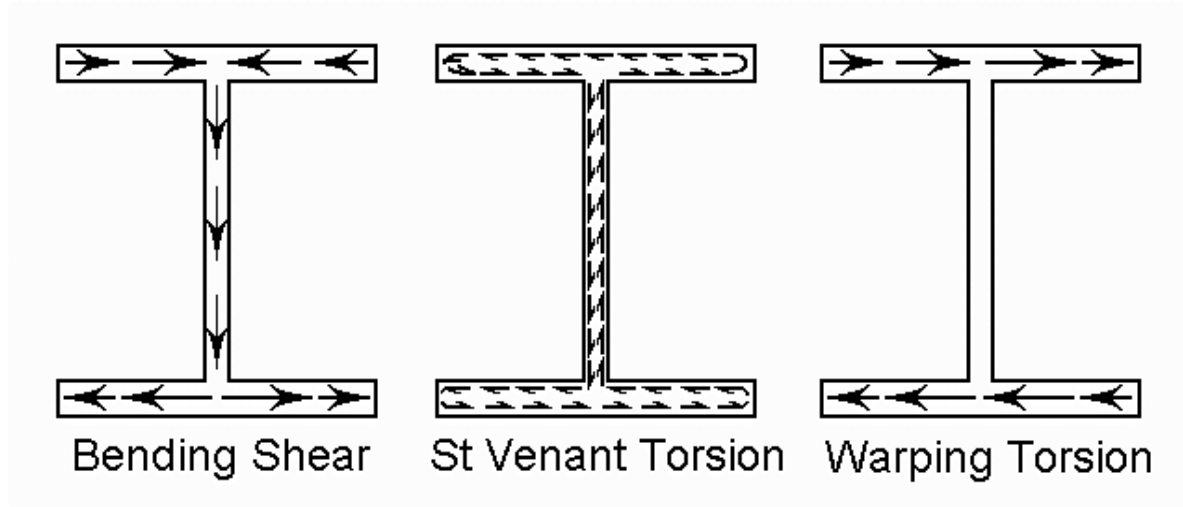
When combined bending and twist of a beam occurs, it is necessary to include all three components of shear flow

Shear Flow due to bending

St Venant Torsion

Warping Torsion

to evaluate the total shear stresses.



The shear flow from bending is calculated in the usual way, as are the St Venant torsion shear stresses. The shear flow from torsion may be calculated as

$$q_{\omega} = -\frac{Q_{\omega}}{C_{\omega}} V_{\omega}$$

However, it is more direct to calculate the shear flow from the fact that

$$W_{\omega} = -EC_{\omega} \frac{d^2\phi}{dx^2} \rightarrow V_{\omega} = -EC_{\omega} \frac{d^3\phi}{dx^3}$$

So,  $q_{\omega} = EQ_{\omega} \frac{d^3\phi}{dx^3}$  provides the easiest calculation, given  $\phi$ .

How do we calculate  $Q_{\omega}$ ? By definition,

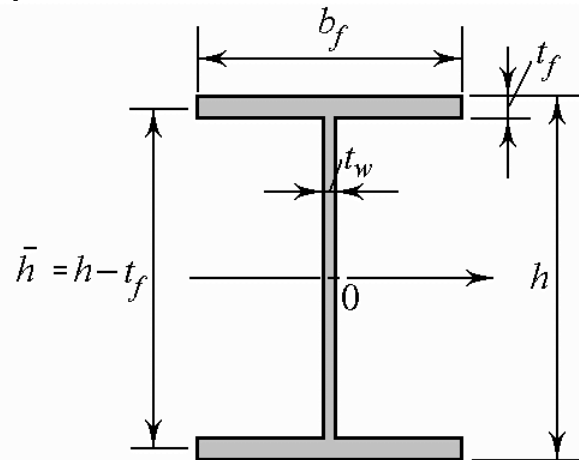
$$Q_{\omega} = \int_0^s 2(\omega_0 - \omega) dA$$

In the integral,  $\omega_0 = \frac{S_\omega}{A}$ , where  $S_\omega = \iint_A 2\omega dA$  was defined

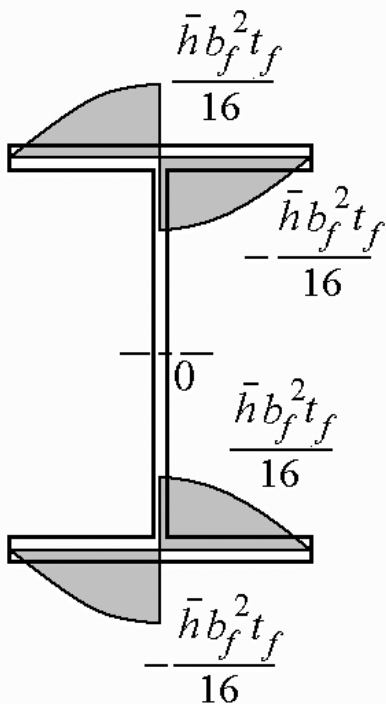
earlier. In these integrals,  $\omega$  is a continuous function of position on the section, starting from 0 at the sectorial origin.

In general, calculating these integrals is messy, but we may often use a simpler approach.

Example: Suppose the above problem involves a W section with properties shown.



Since the beam is doubly symmetrical, the shear center coincides with the centroid. Taking this point as the origin of sectorial areas, and as the sectorial center, the sectorial areas (taken positive when rotated counterclockwise) are as shown below.



The negative and positive  $\omega$  contributions cancel, so  $S_\omega = 0 \rightarrow \omega_0 = 0$ . Integrating the sectorial areas over the flanges, (from the outside edges), the  $Q_\omega$  functions are as shown.

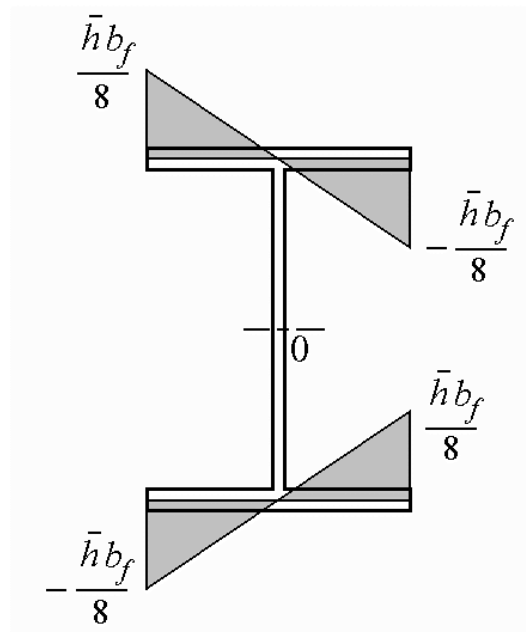
Within the web, the + and - areas from the flanges cancel, so the warping torsion induced shear flow vanishes.

Then,

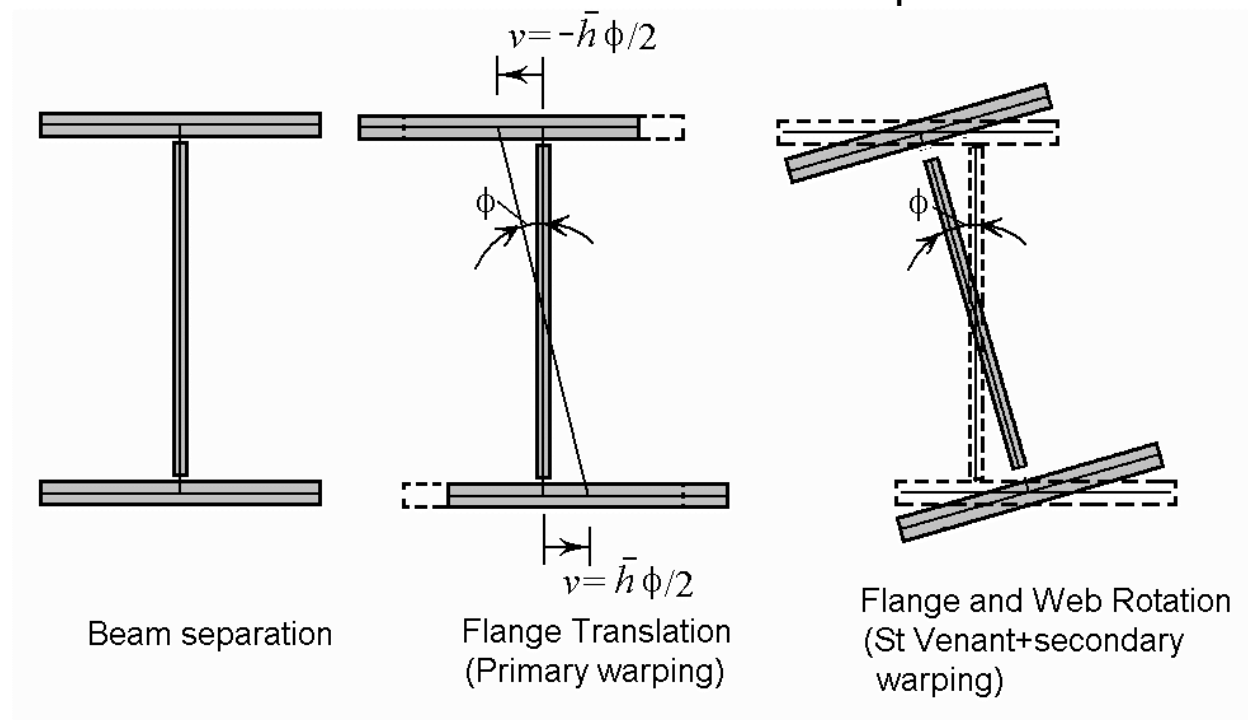
$$q_{\omega} = EQ_{\omega} \frac{d^3 \phi}{dx^3} = \frac{TQ_{\omega}}{2C_{\omega}} \frac{\cosh(\lambda x)}{\cosh(\lambda L/2)}$$

In the above,

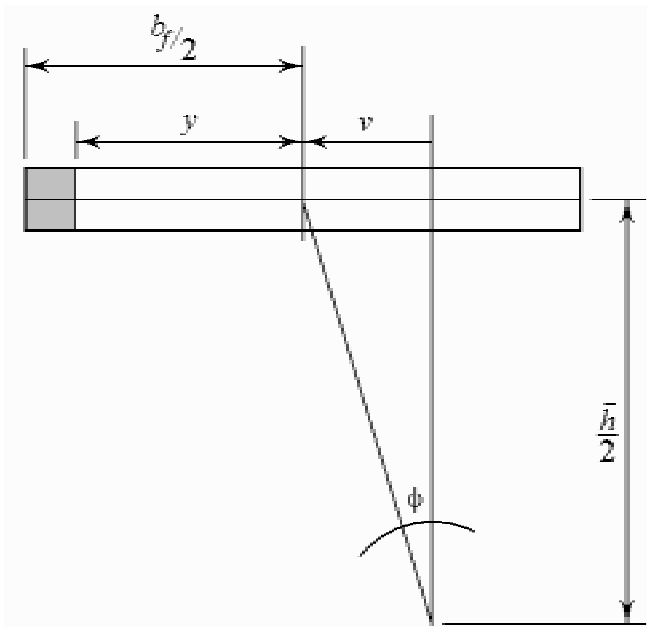
$$C_{\omega} = \frac{\bar{h}^2 b_f^3 t_f}{24} \rightarrow q_{\omega_{\max}}(x) = \frac{3 T}{4 \bar{h} b_f} \frac{\cosh(\lambda x)}{\cosh(\lambda L/2)}$$



The same result can be achieved in a much more intuitive manner. The sketch below illustrates the procedure



The lateral translation  $v$  introduced into the flange by the rotation  $\phi$  causes a bending moment unless  $\phi'$  is constant.



$$M = EI_f v''$$

$$\begin{aligned} \rightarrow V &= EI_f v''' \\ &= EI_f \frac{\bar{h}}{2} \phi'''' \\ &= E \frac{b_f^3 t_f \bar{h}}{12} \phi'''' \end{aligned}$$

This shear produces a torque

$$T_f^{(1)} = V \frac{\bar{h}}{2} = -E \frac{b_f^3 t_f}{12} \left( \frac{\bar{h}}{2} \right)^2 \phi''''$$

A similar torque develops on the bottom flange, so

$$T = T_f^{(1)} + T_f^{(2)} = -E \frac{b_f^3 t_f \bar{h}^2}{24} \phi''''$$

But the warping torsion is given as

$$T_\omega = -EC_\omega \phi'''' \rightarrow C_\omega = \frac{b_f^3 t_f \bar{h}^2}{24}$$

in agreement with the previous result.

Within the top flange, bending introduces a shear flow

$$q_{\omega} = \frac{VQ_f}{I_f}$$

From the above sketch,

$$Q_f = \frac{t_f}{2} \left[ \left( \frac{b_f}{2} \right)^2 - y^2 \right] \quad I_f = \frac{b_f^3 t_f}{12}$$

So

$$q_{\omega} = \frac{\left[ E \frac{b_f^3 t_f}{24} \bar{h} \phi'''' \right] \frac{t_f}{2} \left[ \left( \frac{b_f}{2} \right)^2 - y^2 \right]}{\frac{b_f^3 t_f}{12}}$$

$$= E \frac{\bar{h} t_f}{2} \frac{t_f}{2} \left[ \left( \frac{b_f}{2} \right)^2 - y^2 \right] \phi''''$$

For the beam analyzed previously,

$$\phi''' = -\frac{T}{2EC_\omega} \frac{\cosh(\lambda x)}{\cosh(\lambda L/2)}$$

so

$$\begin{aligned} q_\omega &= \frac{\bar{h}}{2} \frac{t_f}{2} \left[ \left( \frac{b_f}{2} \right)^2 - y^2 \right] \frac{T}{2C_\omega} \frac{\cosh(\lambda x)}{\cosh(\lambda L/2)} \\ &= \frac{\bar{h}}{2} \frac{t_f}{2} \left[ \left( \frac{b_f}{2} \right)^2 - y^2 \right] \frac{24}{b_f^3 t_f \bar{h}^2} \frac{T}{2} \frac{\cosh(\lambda x)}{\cosh(\lambda L/2)} \\ &= \left[ \left( \frac{b_f}{2} \right)^2 - y^2 \right] \frac{3T}{b_f^3 \bar{h}} \frac{\cosh(\lambda x)}{\cosh(\lambda L/2)} \end{aligned}$$

The peak value of warping shear flow occurs at the middle of the flange ( $y=0$ ) at the midspan of the beam ( $x=L/2$ ), where

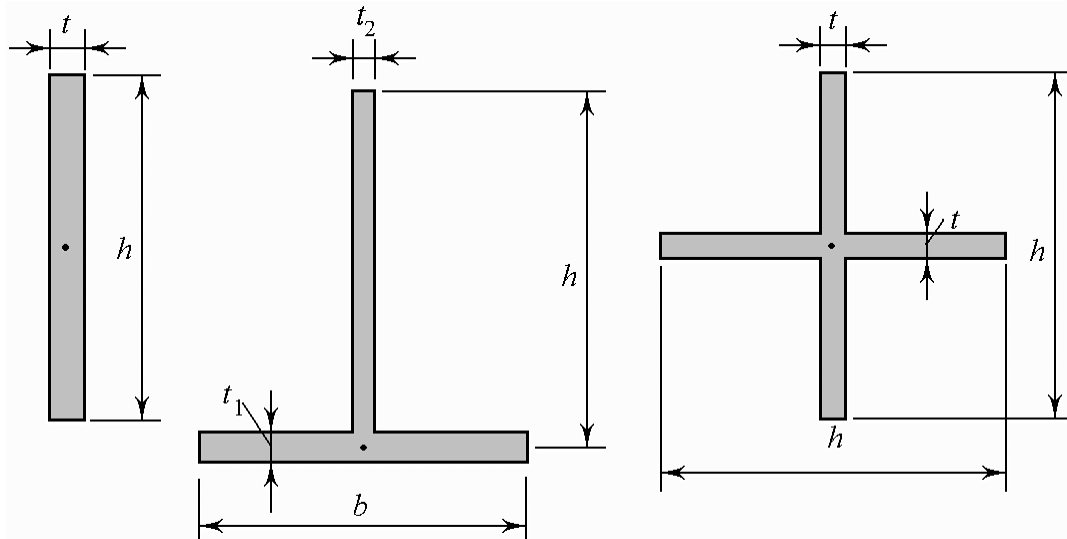
$$q_\omega = \frac{3}{4} \frac{T}{b_f \bar{h}}$$

The third part of the sketch on p. TO.30 re-establishes compatibility by rotating the flanges, and the web about an angle  $\phi$  about their respective centroids. This introduces the St Venant torsion, as well as a secondary warping torsion caused by the plate bending of the individual flange and web plates. We usually ignore the secondary warping *except* for certain torsionally weak sections.



## Secondary Warping:

For certain sections, the *only* resistance to warping is the secondary warping resistance. Several such sections are shown below...

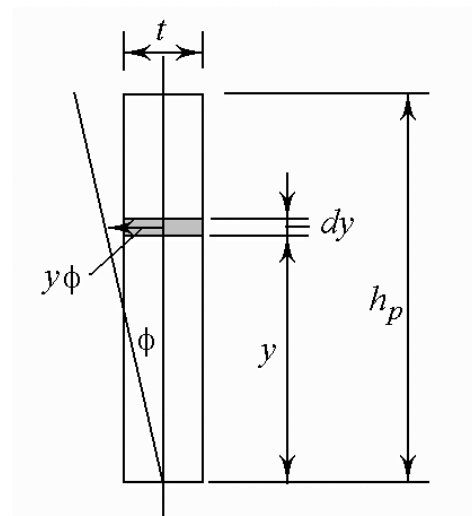


The secondary warping torsional resistance can often be computed for such sections using an analogy to the approach used for the flange above. Consider a plate, rotated about its base through an angle  $\phi$  which varies with  $x$ . At  $y$ ,

$$v'''(y) = y\phi''''$$

$$dV_{\omega_s}(y) = EI_{\omega_s}(y)\phi'''' = E \frac{t^3 dy}{12} y\phi''''$$

$$dT_{\omega_s}(y) = dV_{\omega_s}(y)y = E \frac{t^3 dy}{12} y^2\phi''''$$



Integrating over the height of the plate,

$$T_{\omega s} = \int_0^{h_p} E \frac{t^3 y^2}{12} dy \phi'''' = E \frac{t^3 h_p^3}{36} \phi''''$$

$$\rightarrow C_{\omega s} = \frac{t^3 h_p^3}{36}$$

All of the cases shown above can be constructed by variations of the above. For example, a plate of height  $h$  rotated about its centroid is equivalent to two plates of  $h/2$  rotated about their respective bases. Hence,

$$C_{\omega s} = 2 \frac{t^3 (h/2)^3}{36} = \frac{t^3 h^3}{144}$$

Likewise, the T section has warping stiffness

$$C_{\omega s} = \frac{t_1^3 b^3}{144} + \frac{t_2 h^3}{36}$$

and the cruciform shown has warping stiffness

$$C_{\omega s} = 2 \frac{t^3 h^3}{144} = \frac{t^3 h^3}{72}$$

(You should calculate the corresponding warping stiffness for an angle with unequal legs.)

If the plates are thin, the warping stiffness may be relatively small. This can have an important effect, as we'll see.

## Design Approximations for Torsion

Analysis using the differential equation for torsion is complicated, even for simple loads, so various approximations have been introduced.

Idea: In warping torsion, the flanges act like beams bending laterally.

∴

A (conservative) approximate analysis might be

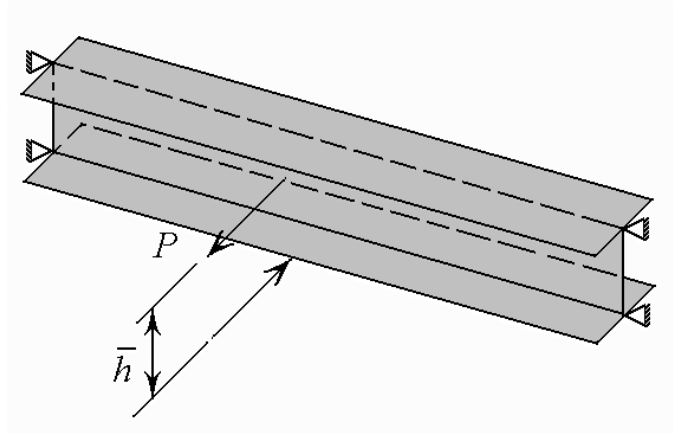
1. convert the torque into couples acting on the flanges by dividing by the beam height.
2. Apply the "equivalent" lateral loads as forces on the flanges.
3. Analyze the flanges as beams under these loads.

Example:

For the beam analyzed before, the torque converts into two lateral loads

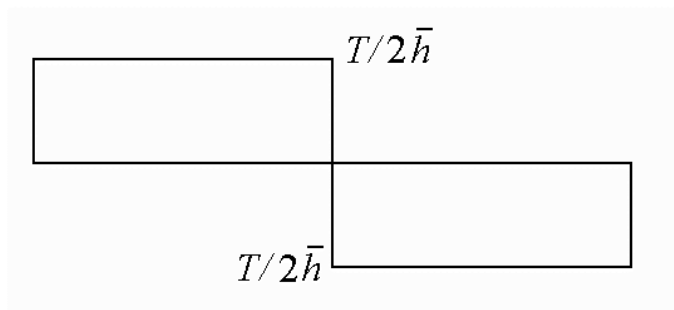
$$P = T / \bar{h}$$

applied at the top and bottom flanges.



The resulting shear is half of this, on each end...

$$V = T / 2\bar{h}$$



The shear flow is thus

$$q_{\omega} \approx VQ_f / I_f = \frac{3 \left[ \left( \frac{b_f}{2} \right)^2 - y^2 \right] T}{\bar{h} b_f^3}$$

The max. shear flow at  $y=0$  is

$$q_{\omega_{\max}} = \frac{3T}{4\bar{h}b_f}$$

which clearly is an upper bound to the actual shear flow.

This approach is often overly conservative, because

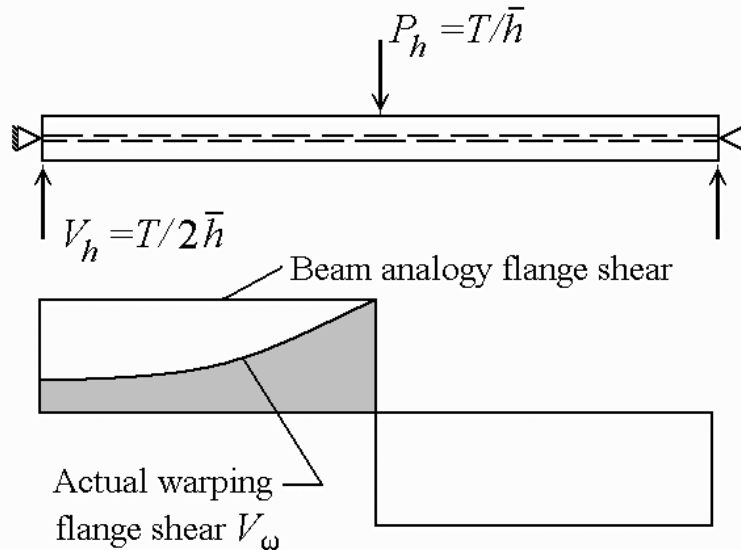
- The normal stresses caused by the bimoment are most important quantitatively, and may be as large as the bending stresses for a beam in combined bending and torsion
- The flange shear flow predicted by the bending analogy will over-estimate the actual warping torsion shear flow over much of the beam. Since, in the same way that

$$V = \frac{dM}{dx} \rightarrow M(x) = M(0) + \int_0^x V(\xi)d\xi$$

We also have

$$V_{\omega} = \frac{dW_{\omega}}{dx} \rightarrow W_{\omega}(x) = W_{\omega}(0) + \int_0^x V_{\omega}(\xi)d\xi$$

For example, by the beam analogy,



The beam analogy yields

$$\rightarrow M_{f_\omega} = \frac{T}{2\bar{h}} \frac{L}{2} \text{ from which } \sigma_{xx_\omega} = \frac{T}{2\bar{h}} \frac{L}{2} \frac{b_f}{I_f}$$

However, the actual shear is

$$V_f = \frac{T}{2\bar{h}} \frac{\cosh(\lambda x)}{\cosh(\lambda L/2)}$$

Hence, the integrated lateral flange moment is

$$M_{f_\omega} = \frac{T}{2\bar{h}} \int_0^{L/2} \frac{\cosh(\lambda x)}{\cosh(\lambda L/2)} dx = \beta \frac{T}{2\bar{h}} \frac{L}{2} < \frac{T}{2\bar{h}} \frac{L}{2}$$

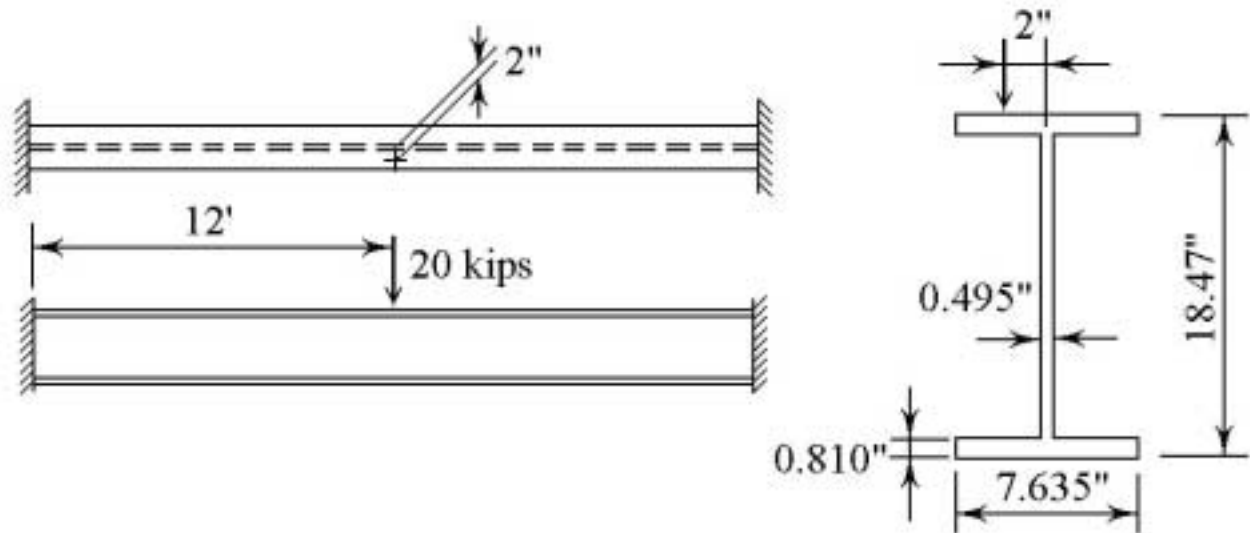
where  $\beta$  is a reduction factor, dependent upon

- $\lambda L$
- The end conditions
- The specifics of the applied torque

S & J give values of  $\beta$  for some common load cases.

Are the stresses caused by torsion large enough to be a problem? Let's see.

Example: A W18x71 beam spanning 24 ft. is loaded with a concentrated load of 20 kips at midspan. The load acts 2" away from the Z axis. The beam is fixed at both ends.



(Variation on S&J example.)

For the beam, the properties are

$$C_{\omega} = \frac{b_f^3 t_f \bar{h}^2}{24}$$

$$= \frac{(7.635'')^3 (.810'')(18.47'' - .81'')^2}{24} = 4,685 \text{ in}^6$$

$$J = \sum \frac{bt^3}{3} = \frac{1}{3} [2(7.635'')(0.810'')^3 + (18.47 - 1.62)(.495'')^3]$$

$$= 3.39 \text{ in}^4$$

$$E/G = 2(1 + \nu) = 2(1 + 0.3) = 2.6$$

∴

$$\lambda = \sqrt{\frac{JG}{EC_{\omega}}} = \sqrt{\frac{3.39}{(4,685)2.6}} = \frac{0.01668}{\text{in.}}$$

For the fixed-fixed case, we can show that

$$\phi = \frac{T}{2JG\lambda} \left[ \lambda x - \sinh(\lambda x) + \frac{(1 - \cosh(\lambda L/2))}{\sinh(\lambda L/2)} (1 - \cosh(\lambda x)) \right]$$

so

$$\phi' = \frac{T}{2JG} \left[ 1 - \cosh(\lambda x) - \frac{(1 - \cosh(\lambda L/2))}{\sinh(\lambda L/2)} \sinh(\lambda x) \right]$$

$$\phi'' = \frac{T\lambda}{2JG} \left[ -\sinh(\lambda x) - \frac{(1 - \cosh(\lambda L/2))}{\sinh(\lambda L/2)} \cosh(\lambda x) \right]$$

(a) Saint-Venant Torsion

$$\tau_{SV} = Gt_f \phi'$$

$$= \frac{Tt}{2J} \left[ 1 - \cosh(\lambda x) + (1 - \cosh(\lambda L/2)) \sinh(\lambda x) / \sinh(\lambda L/2) \right]$$

Here,  $\sinh(\lambda L/2) = 5.477$ ,  $1 - \cosh(\lambda L/2) = -4.567$ , so



$$\tau_{SV} = \frac{(40''k)(0.81'')}{2(3.39 \text{ in}^4)} [1 - \cosh(.01668x) + 0.8338 \sinh(.01668x)]$$

For the fixed-fixed case, the Saint-Venant torsion is zero at both support and centerline, and is maximum at  $L/4$ , where

$$\tau_{SV1} = 2.141 \text{ ksi}$$

(b) Warping Torsion Shear: For the W section, following the above reasoning,

$$\tau_{\omega_{\max}} = E \frac{b_f^2 \bar{h}}{16} \phi''''$$

From the solution,

$$\phi'''' = \frac{T\lambda^2}{2JG} \left[ -\cosh(\lambda x) - \frac{(1 - \cosh(\lambda L/2))}{\sinh(\lambda L/2)} \sinh(\lambda x) \right]$$

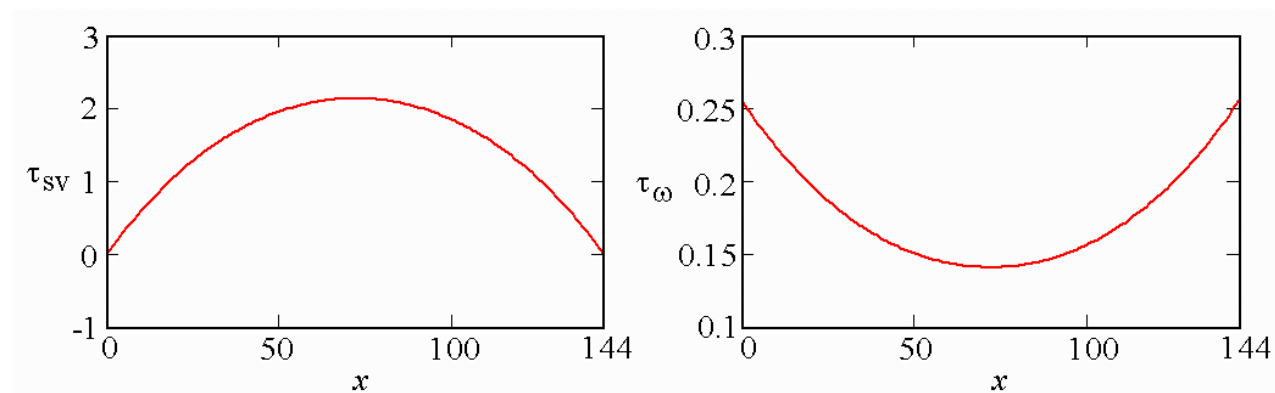
so

$$\begin{aligned} \tau_{\omega_{\max}} &= \frac{b_f^2 \bar{h} T}{32C_\omega} \left[ -\cosh(\lambda x) - \frac{1 - \cosh(\lambda L/2)}{\sinh(\lambda L/2)} \sinh(\lambda x) \right] \\ &= 0.2556 [-\cosh(.01668x) + 0.8338 \sinh(.01668x)] \end{aligned}$$

$\tau_{\omega_{\max}}$  reaches its max. amplitude at the end, and at the centerline, where the warping restraint is greatest. There

$$|\tau_\omega(0)| = |\tau_\omega(L/2)| = 0.2556$$

It is smallest at the quarter points where warping restraint is least.



(c) Lateral bending stresses introduced by warping torsion

The bending stress caused by warping is given by

$$\sigma_{xx_\omega} = -\omega E \frac{d^2\phi}{dx^2}$$

From the above analysis, the largest magnitude stress occurs at the flange tips, where  $\omega = b_f \bar{h} / 4$ . Thus,

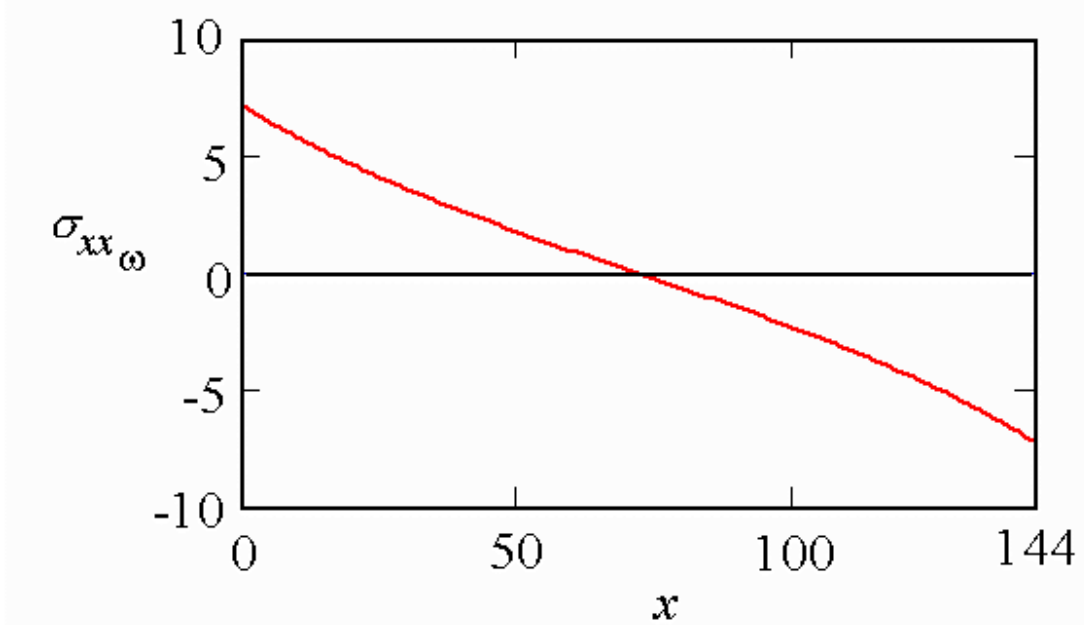
$$\begin{aligned} |\sigma_{xx_\omega}| &= \frac{b_f \bar{h}}{4} E \phi'' \\ &= \frac{b_f \bar{h}}{4} \frac{E \lambda T}{G 2J} [-\sinh(\lambda x) - (1 - \cosh(\lambda L / 2)) \cosh(\lambda x) / \sinh(\lambda L / 2)] \end{aligned}$$

at the flange tips

Substituting the numerical values leads to

$$\begin{aligned}\sigma_{xx\omega} &= \frac{(7.635'')(17.66'')}{4} (2.6) \frac{(0.01668/\text{in.})(40''\text{k})}{2(3.39\text{ in}^4)} \\ &\quad [-\sinh(.01668x) + 0.8338\cosh(.01668x)] \\ &= 8.625[-\sinh(.01668x) + 0.8338\cosh(.01668x)]\end{aligned}$$

The maximum values of  $\sigma_{xx\omega}$  occur at the ends, *and at midspan*, where  $|\sigma_{xx\omega}| = 7.192$  ksi (compressive on one side of the flange, tensile on the other).



The maximum normal stress on one edge of the flange is shown over a half beam length. Maximum warping restraint exists at the support, and at midspan.

(d) Bending analogy... For this case

$$\bar{h} = 17.66", T = 40" \text{ k} \rightarrow P_f = \frac{40}{17.66} = 2.265 \text{ kips}$$

For a fixed-fixed beam, max moments are

$$M = \frac{P_f L}{8} = \frac{(2.265 \text{ kips})(288")}{8} = 81.54" \text{ k}$$

$$\text{Then, if } S_{fl} = \frac{t_f b_f^2}{6} = \frac{(0.81")(7.635")^2}{6} = 7.8696 \text{ in}^3$$

$$\sigma_{xx_{equiv.}} = \frac{81.54" \text{ k}}{7.8696 \text{ in}^3} = 10.361 \text{ ksi}$$

As noted above, this is overly conservative.  
From table 8.6.2, the interpolated  $\beta$  value is

$$\beta = (0.8)(0.68) + (0.2)(0.76) = 0.696$$

Then,

$$\sigma_{xx_{\omega}} \approx 0.696(10.361 \text{ ksi}) = 7.211 \text{ ksi}$$

which is pretty close to the maximum value obtained from the closed form solution.

Observations:

- The Saint-Venant torsion shear stresses are somewhat larger than the warping torsion shear stresses, but both are relatively small.
- The warping normal stresses may be significant.

(e) To see how significant, let's compute the bending stresses...

$$M = \frac{PL}{8} = \frac{(20 \text{ kips})(288'')}{8} = 720'' \text{ k}$$

$$S_x = 127 \text{ in}^3$$

$$\sigma_{xx \text{ bending}} = 720'' \text{ k} / 127 \text{ in}^3 = 5.67 \text{ ksi}$$

In this case, the warping normal stresses are larger than the bending stresses!

## 5. Other Approaches:

Calculating  $\phi$  may be simplified by tabulated solutions for certain idealized boundary conditions, or a matrix stiffness equation

$$\begin{Bmatrix} T_1 \\ W_{\omega_1} \\ T_2 \\ W_{\omega_2} \end{Bmatrix} = \frac{JG}{\Delta} \begin{bmatrix} a_1 & a_2 & -a_1 & a_2 \\ a_2 & a_3 & -a_2 & a_4 \\ -a_1 & -a_2 & a_1 & -a_2 \\ a_2 & a_4 & -a_2 & a_3 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_1' \\ \phi_2 \\ \phi_2' \end{Bmatrix}$$

where

$$\Delta = \lambda^2 L \sinh(\lambda L) - 2\lambda \cosh(\lambda L) + 2\lambda$$

$$a_1 = \lambda^2 \sinh(\lambda L) \qquad a_2 = \lambda[\cosh(\lambda L) - 1]$$

$$a_3 = \lambda L \cosh(\lambda L) - \sinh(\lambda L) \qquad a_4 = \sinh(\lambda L) - \lambda L$$

may be useful.

This stiffness matrix may be used to replace the 2x2 torsional stiffness submatrix

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{JG}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix}$$

commonly used in the beam (or frame) equations, and incorporates the end conditions commonly encountered.

This approach has the advantage that it can produce reasonable torsional stiffnesses for applications involving torsional moments caused by eccentrically connected beams (See S&J, section 8.7)

The associated solution

$$\phi(x) = f_1(x)\phi_1 + f_2(x)\phi_1' + f_3(x)\phi_2 + f_4(x)\phi_2'$$

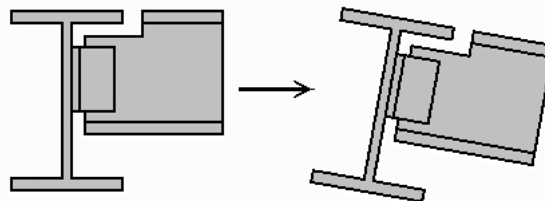
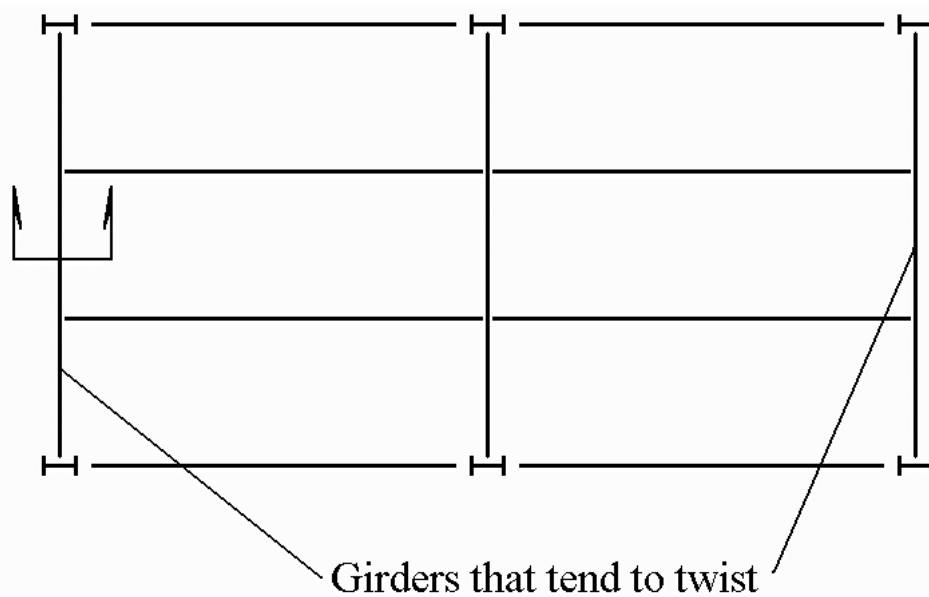
can then be used to obtain all derivatives of  $\phi$  needed to calculate the shear flow, where

$$\begin{aligned} f_1 &= [a_1(L-x) - a_2(1 + \cosh(\lambda x)) + a_1 \sinh(\lambda x)/\lambda] / \Delta \\ f_2 &= [-a_2 x + a_3(1 - \cosh(\lambda x)) + (a_1 L - a_2) \sinh(\lambda x)/\lambda] / \Delta \\ f_3 &= [a_1(x - \sinh(\lambda x)/\lambda) - a_2(1 - \cosh(\lambda x))] / \Delta \\ f_4 &= [a_4(1 - \cosh(\lambda x)) - a_2(x - \sinh(\lambda x)/\lambda)] / \Delta \end{aligned}$$

are the "shape functions" for the problem. Distributed torques can also be accommodated.

S&J suggest that the best approach for determining the applied torque is one that establishes compatibility between the twisting rotation  $\phi$  of the beam supporting a torque producing member, and the end bending rotation of the attached element. They note that

- Applying the shear at the face of the web underestimates the torsion
- Applying the shear at the centerline of the connection overestimates the torsion
- Applying the shear so as to achieve rotational continuity with the end of the beam framing in tends to be about right.





## 6. LRFD for Torsional Moments

Assumption: Torsion stresses add to bending stresses, but lateral torsional instability is prevented.

i.e. lateral restraint exists, but laterally restraining members may induce self-limiting torsion before restraint becomes effective.

Other problems exist as well, for which torsional rotation may not be self limited.

AISC Philosophy: (consistent with ASD)

- Assume the limit state occurs when yielding is reached.  
∴ Elastic analysis is applicable.
- Compute factored stresses using  $w_D, w_L$ , etc, e.g.  
 $w_u = 1.2w_D + 1.6w_L, \dots$
- Once the factored design moment and factored design lateral flange moments are known, the stresses are superpositioned with the limit state. Formally,

$$\frac{M_{b_z}}{S_z} + \frac{M_{\omega}}{S_{\omega}} \leq \phi_b F_y$$

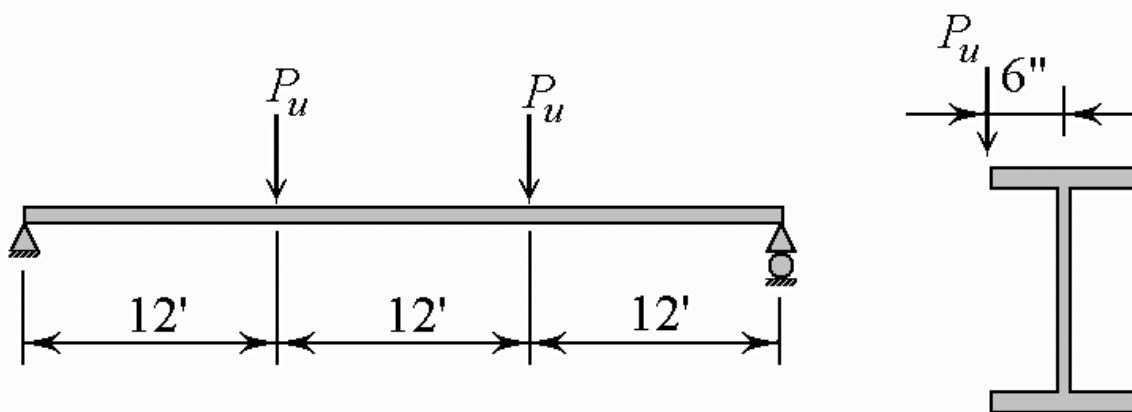
or, in terms of the equivalent lateral flange moment

$$\frac{M_{b_z}}{S_z} + \frac{2M_f}{S_y} \leq \phi_b F_y$$

In the above,  $\phi_b = 0.9$ =resistance factor.

- Since  $\lambda$  is a section property that influences the  $M_{\omega}$  term, through the  $\beta$  factor, or through the d.e. solution, iterative solution is necessary.
  - Guess a value of  $\lambda$  in the right approximate range.
  - Use the corresponding value of  $\lambda L$ , to determine the value of  $M_{\omega}$  to be used with the design bending moment.
  - Use the bounds to estimate a section size.
  - Use the section size to update the estimated value of  $\lambda$ , etc.

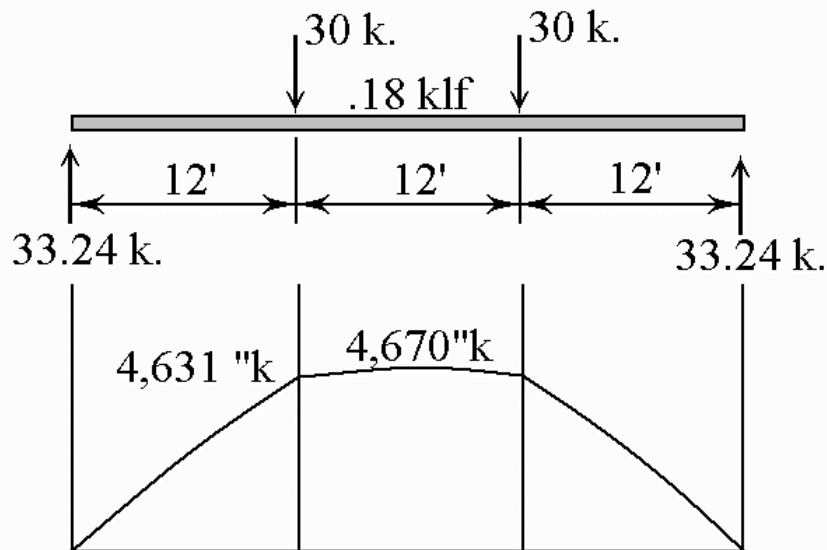
Example: An A36 beam with torsionally simply supported ends must carry two 20 kip loads acting at an eccentricity of 6" (5 Kips DL, 15 Kips LL) (Variation on S&J example)



$$P_u = 1.2(5 \text{ kips}) + 1.6(15 \text{ kips}) = 30 \text{ kips}$$

$$\text{Guess } w_{beam} = 150 \text{ plf} \rightarrow w_u = 1.2(150) = 180 \text{ plf}$$

Bending moments:



Torsion:

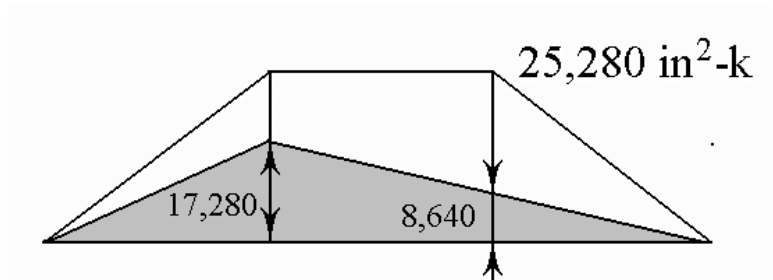
$$T_u = (6")(30 \text{ kips}) = 180" \text{ k.}$$

Using beam analogy

$$P_f = T_u / h = 180" \text{ k} / \bar{h}$$

$$M_f = P_f L / 3 = (180" \text{ k})(144") / \bar{h}$$

$$= 25,920 \text{ in}^2 \text{ k} / \bar{h}$$



The  $\beta$  reduction factor depends upon  $\lambda L$ . Guess  $\lambda = 0.015$ , (for a majority of W sections,  $.01 < \lambda < .02$ ) Thus,

$$\lambda L = .015(432") = 6.48 \rightarrow \beta \approx 0.36$$

The  $\beta$  value for the torques in the middle of the beam aren't exactly known, but the estimates tend to be somewhat conservative.

Guessing a 14" beam, the reduced flange moment is

$$M_f = 0.36(25,920 \text{ in}^2 \cdot \text{k}) / 14" = 667" \text{ k}$$

For a midsized W14,  
 $S_y / S_z \approx 2.6$ , taking both flanges  
 into account, so

$$\frac{4670" \text{ k}}{S_y} + \frac{667" \text{ k} \times 2}{S_y / 2.6} \leq 0.9(36 \text{ ksi})$$

$$\rightarrow S_y \geq 251 \text{ in}^3$$

Try W14x159. For this section,  $S_y / S_z = 2.64$ , so the guess is o.k. for now. Also, for this section

$$\frac{1}{\lambda} = \sqrt{\frac{EC_\omega}{GJ}} = 68.2 \quad (\text{p. 1-148 of AISC-LRFD})$$

$$\text{so } \lambda L = 432" / 68.2 = 6.33$$

This implies a revised  $\beta = 0.37$

Now,

$$M_f = 0.37(25,920)/13.79" = 695" \text{ k}$$

So

$$S_y \geq \frac{4,670" \text{ k}}{32.4 \text{ ksi}} + \frac{(695" \text{ k})(2)}{32.4/2.64} = 257 \text{ in}^3$$

The W14x159 is not quite big enough, since  $S_y = 254 \text{ in}^3$ . (actually, it's probably o.k., in view of the approximations in calculating  $\beta$ , but without a more precise calculation, we can't be sure.)

Try a W14x176 Revise the moment upward to 4,720" k to account for the extra dead load.

From the tables,

$$\lambda = 1/62.9 \rightarrow \lambda L = 432/62.9 = 6.86$$

$$S_y =$$

$$S_y / S_z = 2.63$$

Revised  $\beta = 0.35$

$$\bar{h} = 13.91"$$

$$M_f = .35(25,920)/13.91 = 652" \text{ k}$$

$$\frac{4720}{281} + \frac{652 \times 2}{281/2.63} = 29 \text{ ksi} < 32.4 \text{ ksi}$$

o.k. A W14x176 is more than adequate here

## 7. Presence of Axial Loads (Torsional Instability)

In presence of axial loads, or under certain lateral load conditions, instability can occur. In this case, formulation in the displaced state is necessary.

The relevant displacements are  $v$ ,  $w$  and  $\phi$ . So, the first equation (for  $u_0$ ) is not changed.

Formulation is complicated, (as is the result) so, we'll just present the resulting equations for  $v$ ,  $w$  and  $\phi$ .

The complete equations are nonlinear. A linear version which omits the non-linear terms is presented here.

$$\begin{aligned}
 & E(I_{yy}I_{zz} - I_{yz}^2) \frac{d^4v}{dx^4} - I_{yy}N_x \frac{d^2v}{dx^2} + I_{yz}N_x \frac{d^2w}{dx^2} \\
 & + [I_{yy}M_{ly} + I_{yz}M_{lz} - N_x(e_z I_{yy} + e_y I_{yz})] \frac{d^2\phi}{dx^2} \\
 & + (I_{yy}V_z + I_{yz}V_y) \frac{d\phi}{dx} = q_y I_y - q_z I_{yz}
 \end{aligned}$$

$$\begin{aligned}
& E(I_{yy}I_{zz} - I_{yz}^2) \frac{d^4 w}{dx^4} - I_{zz} N_x \frac{d^2 w}{dx^2} + I_{yz} N_x \frac{d^2 v}{dx^2} \\
& + [-I_{zz} M_{lz} - I_{yz} M_{ly} + N_x (e_y I_{zz} + e_z I_{yz})] \frac{d^2 \phi}{dx^2} \\
& - (I_{zz} V_y - I_{yz} V_z) \frac{d\phi}{dx} = q_z I_{zz} - q_y I_{yz} \\
& EC_\omega \frac{d^4 \phi}{dx^4} - \left( GJ + \frac{I_E}{A} N_x + C_z M_{lz} + C_y M_{ly} + \frac{H_\omega W_\omega}{C_\omega} \right) \frac{d^2 \phi}{dx^2} \\
& + (M_{ly} - e_z N_x) \frac{d^2 v}{dx^2} - (M_{lz} - e_y N_x) \frac{d^2 w}{dx^2} + V_z \frac{dv}{dx} - V_y \frac{dw}{dx} \\
& - \left( C_z V_y + C_y V_z + \frac{H_\omega V_\omega}{C_\omega} \right) \frac{d\phi}{dx} = t(x)
\end{aligned}$$

In these equations, a number of additional quantities are to be defined.

$M_{ly}, M_{lz}$  = Moments caused by transverse loads only

$I_E = I_{yy} + I_{zz} + (e_y^2 + e_z^2)A$  = polar moment of inertia about shear center

Three new geometrical quantities are

$$H_y = \iint_A z(y^2 + z^2) dA$$

$$H_z = \iint_A y(y^2 + z^2) dA$$

$$H_\omega = \iint_A 2(\omega_0 - \omega)(y^2 + z^2) dA$$

From these,

$$C_y = \frac{I_{zz}H_y - I_{yz}H_z}{I_{yy}I_{zz} - I_{yz}^2} - 2e_z$$

$$C_z = \frac{I_{yy}H_z - I_{yz}H_y}{I_{yy}I_{zz} - I_{yz}^2} - 2e_y$$

The equations are three simultaneous equations in three unknowns. Although the geometrical properties are constant, the load coefficients  $M_{ly}, M_{lz}, V_y, V_z, W_\omega, V_\omega$  all vary with  $x$ .

∴ The d.e. coefficients are variable.

Often, the d.e.s can be simplified, by taking advantage of special properties of different classes of problems.

We will want to do this in practice, since the original set of equations is unwieldy.



First, assume that the y-z coordinates are principal coordinate axes. Then  $I_{yz} = 0$ , and the eqns reduce to

$$EI_{zz} \frac{d^4 v}{dx^4} - N_x \frac{d^2 v}{dx^2} + (M_{ly} - N_x e_z) \frac{d^2 \phi}{dx^2} + V_{zy} \frac{d\phi}{dx} = q_y$$

$$EI_{yy} \frac{d^4 w}{dx^4} - N_x \frac{d^2 w}{dx^2} + (-M_{lz} + N_x e_y) \frac{d^2 \phi}{dx^2} - V_y \frac{d\phi}{dx} = q_z$$

$$EC_{\omega} \frac{d^4 \phi}{dx^4} - \left( GJ + \frac{I_E}{A} N_x + C_z M_{lz} + C_y M_{ly} + \frac{H_{\omega}}{C_{\omega}} W_{\omega} \right) \frac{d^2 \phi}{dx^2} \\ + (M_{ly} - e_z N_x) \frac{d^2 v}{dx^2} - (M_{lz} - e_y N_x) \frac{d^2 w}{dx^2} + V_z \frac{dv}{dx} - V_y \frac{dw}{dx} \\ - \left( C_z V_y + C_y V_z + \frac{H_{\omega}}{C_{\omega}} V_{\omega} \right) \frac{d\phi}{dx} = t(x)$$

If, in addition, the transverse loads  $q_y, q_z$  are zero, and the transverse moments are constant, then the shears  $V_y, V_z$  are also zero, and the additional simplification

$$EI_{zz} \frac{d^4 v}{dx^4} - N_x \frac{d^2 v}{dx^2} + (M_{ly} - N_x e_z) \frac{d^2 \phi}{dx^2} = 0$$

$$EI_{yy} \frac{d^4 w}{dx^4} - N_x \frac{d^2 w}{dx^2} + (-M_{lz} + N_x e_y) \frac{d^2 \phi}{dx^2} = 0$$

$$EC_{\omega} \frac{d^4 \phi}{dx^4} - \left( GJ + \frac{I_E}{A} N_x + C_z M_{l_z} + C_y M_{l_y} \right) \frac{d^2 \phi}{dx^2} - \frac{H_{\omega}}{C_{\omega}} \frac{d}{dx} \left( W_{\omega} \frac{d\phi}{dx} \right) + (M_{l_y} - e_z N_x) \frac{d^2 v}{dx^2} - (M_{l_z} - e_y N_x) \frac{d^2 w}{dx^2} = t(x)$$

is possible.

Now, assuming that the bar is subjected only to an axial *compressive* load through the centroid, the moments due to transverse loads are zero, the applied torque vanishes, and if the loading is applied so that the bimoment is zero, the equations become

$$EI_{zz} \frac{d^4 v}{dx^4} + N_x \frac{d^2 v}{dx^2} + N_x e_z \frac{d^2 \phi}{dx^2} = 0$$

$$EI_{yy} \frac{d^4 w}{dx^4} + N_x \frac{d^2 w}{dx^2} - N_x e_y \frac{d^2 \phi}{dx^2} = 0$$

$$EC_{\omega} \frac{d^4 \phi}{dx^4} - \left( GJ - \frac{I_E}{A} N_x \right) \frac{d^2 \phi}{dx^2} + e_z N_x \frac{d^2 v}{dx^2} - e_y N_x \frac{d^2 w}{dx^2} = 0$$

These equations represent a generalization of the elastic column buckling problem. (Note sign change in  $N_x$  because of compression.)

Assume a pinned end column for which the appropriate boundary conditions are

$$v = 0 \quad w = 0 \quad \phi = 0$$

$$v'' = 0 \quad w'' = 0 \quad \phi'' = 0$$

Then a solution of the form

$$v = v_0 \sin(\pi x / L)$$

$$w = w_0 \sin(\pi x / L)$$

$$\phi = \phi_0 \sin(\pi x / L)$$

where  $v_0, w_0, \phi_0$  are the values of  $v, w, \phi$  at midspan, will satisfy the resulting equations provided

$$(EI_{zz} \frac{\pi^2}{L^2} - N_x)v_0 - N_x e_z \phi_0 = 0$$

$$(EI_{yy} \frac{\pi^2}{L^2} - N_x)w_0 + N_x e_y \phi_0 = 0$$

$$-e_z N_x v_0 + e_y N_x w_0 + \left[ EC_\omega \frac{\pi^2}{L^2} + GJ - \frac{I_E}{A} N_x \right] \phi_0 = 0$$

or, in matrix form

$$\begin{bmatrix} EI_{zz} \frac{\pi^2}{L^2} - N_x & 0 & -N_x e_z \\ 0 & EI_{yy} \frac{\pi^2}{L^2} - N_x & N_x e_y \\ -N_x e_z & N_x e_y & EC_\omega \frac{\pi^2}{L^2} + GJ - \frac{I_E}{A} N_x \end{bmatrix} \begin{Bmatrix} v_0 \\ w_0 \\ \phi_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Non-trivial solutions exist iff the determinant of coefficients is zero. Rather than solve for the general case, let's consider two special cases.

(a) The section is doubly symmetric. Then  $e_y = e_z = 0$ , and the equations reduce to

$$\begin{bmatrix} EI_{zz} \frac{\pi^2}{L^2} - N_x & 0 & 0 \\ 0 & EI_{yy} \frac{\pi^2}{L^2} - N_x & 0 \\ 0 & 0 & EC_{\omega} \frac{\pi^2}{L^2} + GJ - \frac{I_E}{A} N_x \end{bmatrix} \begin{Bmatrix} v_0 \\ w_0 \\ \phi_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The corresponding determinantal equation

$$\left( EI_{zz} \frac{\pi^2}{L^2} - N_x \right) \left( EI_{yy} \frac{\pi^2}{L^2} - N_x \right) \left( EC_{\omega} \frac{\pi^2}{L^2} + GJ - \frac{I_E}{A} N_x \right) = 0$$

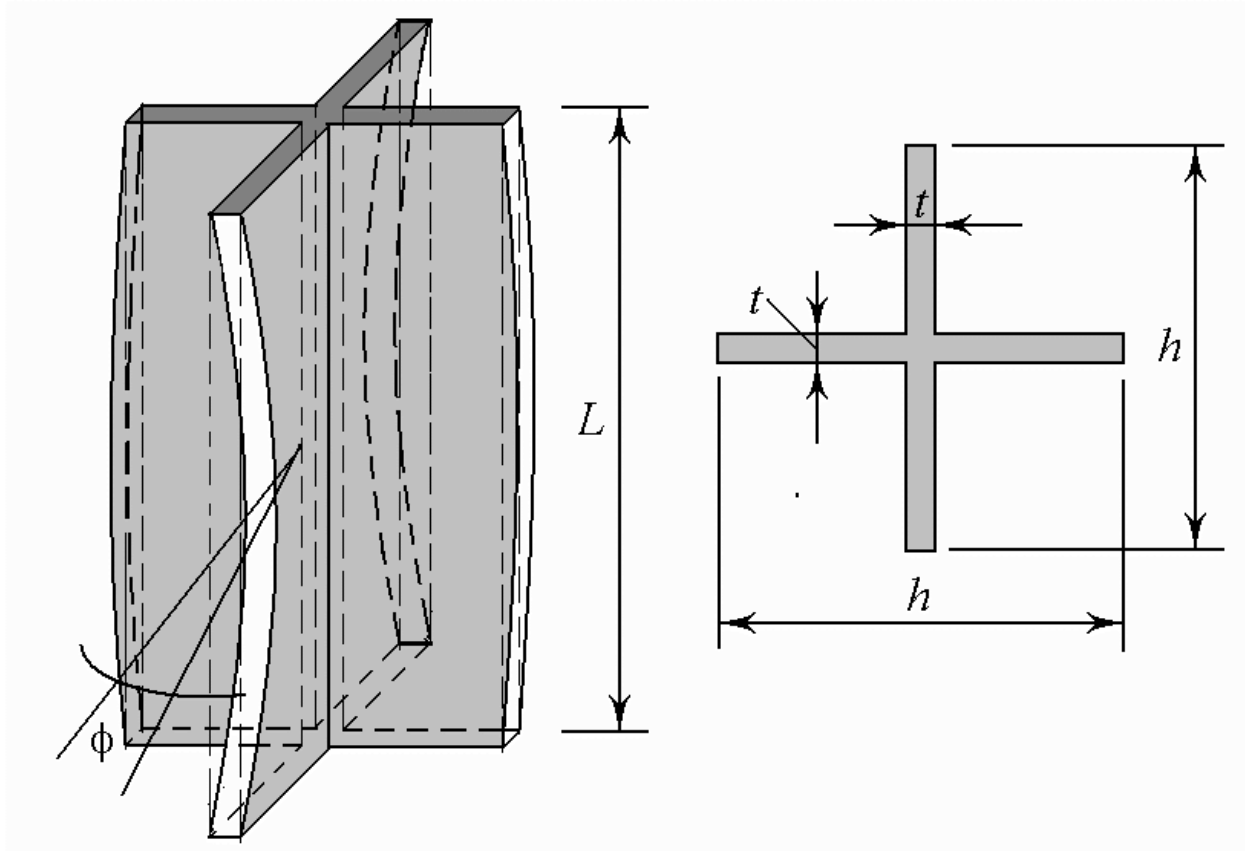
is already factored, and has the three solutions

$$N_x = EI_{zz} \frac{\pi^2}{L^2} = P_{E_{zz}} \quad (\text{buckling about z axis})$$

$$N_x = EI_{yy} \frac{\pi^2}{L^2} = P_{E_{yy}} \quad (\text{buckling about y axis})$$

$$N_x = \frac{A}{I_E} \left( EC_{\omega} \frac{\pi^2}{L^2} + GJ \right) = P_T \quad (\text{Torsional, or twist buckling})$$

The 3<sup>rd</sup> solution corresponds to a new mode of buckling, shown below, for a cruciform section.



Dividing by  $A$  and noting that  $I_E = I_p$  for a doubly symmetrical section yields the critical stress

$$\sigma_{xx_{cr}} = -\frac{1}{I_p} \left( EC_{\omega} \frac{\pi^2}{L^2} + GJ \right)$$

If this critical stress is less than the minor axis Euler stress, then torsional buckling will precede Euler buckling of the section, and must be accounted for in design.

Example: Consider an equal leg cruciform section with  $h=6"$ ,  $t=1/4"$ .

Section properties:

$$I_{yy} = I_{zz} = \frac{th(h^2 + t^2)}{12} = \frac{(6")(0.25")[(6")^2 + (0.25")^2]}{12} = 4.508 \text{ in}^4$$

$$I_p = I_{yy} + I_{zz} = 9.016 \text{ in}^4$$

$$A = 2th - t^2 = 2(6")(0.25") - (0.25")^2 = 2.938 \text{ in}^2 \rightarrow r_y = r_z = 1.239"$$

$$C_{\omega} = \frac{t^3 h^3}{72} = \frac{(0.25")^3 (6")^3}{72} = 0.04688 \text{ in}^6$$

$$J = 4 \frac{(h/2)t^3}{3} = \frac{2ht^3}{3} = \frac{2(6")(0.25")^3}{3} = 0.0625 \text{ in}^4$$

For Euler buckling, both axes have the same  $I$ , so

$$\sigma_{xxcr}^{(Euler)} = \frac{\pi^2 E}{(L/r)^2} = \frac{\pi^2 (30,000)}{L^2} (1.239")^2 = \frac{454,530}{L^2}$$

For torsional buckling,

$$\begin{aligned}
 \sigma_{xx_{cr}}^{(Torsion)} &= \frac{1}{I_p} \left( EC_{\omega} \frac{\pi^2}{L^2} + GJ \right) \\
 &= \frac{1}{9.016} [(30,000)(0.04688) \frac{\pi^2}{L^2} + (11,540)(0.0625)] \\
 &= \frac{1,540}{L^2} + 80
 \end{aligned}$$

(The  $GJ$  term dominates the torsional buckling equation unless the column is very short.)

Torsion controls if  $\sigma_{xx_{cr}}^{(Torsion)} < \sigma_{xx_{cr}}^{(Euler)}$ , so

$$\frac{1540}{L^2} + 80 < \frac{454,530}{L^2} \rightarrow L < 75.2"$$

- In a symmetrical section, torsional buckling tends to control only for short columns.
  - The thinner the walls of the column, relative to the plate widths (Thin plates) the more likely it is that torsional buckling will control. (Increasing the wall thickness to 3/8" in the above calculations decreases the length column for which torsion controls to 50".)
  - It may be feasible to ignore the warping torsion term in the above in the practical range of interest.
  - For the torsion buckling, it is appropriate to replace  $E$  with  $E/(1 - \nu^2)$  consistent with dominant plate action.
- (b) The section is singly symmetrical. If the  $z$  axis is an axis of symmetry, then  $e_y = 0$ , and

$$\begin{bmatrix} EI_{zz} \frac{\pi^2}{L^2} - N_x & 0 & -N_x e_z \\ 0 & EI_{yy} \frac{\pi^2}{L^2} - N_x & 0 \\ -N_x e_z & 0 & EC_\omega \frac{\pi^2}{L^2} + GJ - \frac{I_E}{A} N_x \end{bmatrix} \begin{Bmatrix} v_0 \\ w_0 \\ \phi_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

From which, the determinantal equation is

$$\left( EI_{yy} \frac{\pi^2}{L^2} - N_x \right) \left[ \left( EI_{zz} \frac{\pi^2}{L^2} - N_x \right) \left( EC_\omega \frac{\pi^2}{L^2} + GJ - \frac{I_E}{A} N_x \right) - N_x^2 e_z^2 \right] = 0$$

Before solving this problem, it is useful to substitute the relations

$$P_{E_{zz}} = EI_{zz} \frac{\pi^2}{L^2} \quad P_{E_{yy}} = EI_{yy} \frac{\pi^2}{L^2} \quad \frac{I_E}{A} P_T = EC_\omega \frac{\pi^2}{L^2} + GJ$$

Then, the determinantal eqn. takes the form

$$\left( P_{E_{yy}} - N_x \right) \left[ \frac{I_E}{A} (P_{E_{zz}} - N_x) (P_T - N_x) - N_x^2 e_z^2 \right] = 0$$

This is an important case, as T beams, one section commonly used for compression elements (e.g. truss members) is singly symmetrical.

The leading factor is just Euler column buckling in the plane of symmetry.

The other factor leads to a coupled flexural-torsional buckling mode. The formal solution is



$$N_x = \frac{(P_{E_{zz}} + P_T) - \sqrt{(P_{E_{zz}} + P_T)^2 - 4\left(1 - \frac{Ae_z^2}{I_E}\right)P_{E_{zz}}P_T}}{2\left(1 - \frac{Ae_z^2}{I_E}\right)}$$

Typically, this critical load will be smaller than either the twist buckling load, or the flexural buckling mode.

Dividing the equation by the area permits the relationship to be written in terms of the critical stresses for the individual buckling modes.

$$\sigma_{xx_{cr}}^{(FT)} = \frac{(\sigma_{xx_{cr}}^{(E)} + \sigma_{xx_{cr}}^{(T)}) - \sqrt{(\sigma_{xx_{cr}}^{(E)} + \sigma_{xx_{cr}}^{(T)})^2 - 4\left(1 - \frac{Ae_z^2}{I_E}\right)\sigma_{xx_{cr}}^{(E)}\sigma_{xx_{cr}}^{(T)}}}{2\left(1 - \frac{Ae_z^2}{I_E}\right)}$$

where

$$\sigma_{xx_{cr}}^{(E)} = E \frac{\pi^2}{(L/r_y)^2} \quad \sigma_{xx_{cr}}^{(T)} = \frac{E}{I_E} \left( C_\omega \frac{\pi^2}{L^2} + \frac{J}{2(1+\nu)} \right)$$

One can use the latter equation to define an *equivalent radius of gyration* for torsional buckling. Equating

$$\frac{\pi^2 E}{(L/r_E)^2} = \frac{E}{I_E} \left( C_\omega \frac{\pi^2}{L^2} + \frac{J}{2(1+\nu)} \right)$$

yields the resulting equivalent  $r$

$$r_E = \sqrt{\frac{1}{I_E} \left( C_\omega + \frac{JL^2}{2\pi^2(1+\nu)} \right)}$$

Example: Consider a WT12x27.5:

$$A = 8.10 \text{ in}^2$$

$$I_{yy} = 117 \text{ in}^4 \quad I_{zz} = 14.5 \text{ in}^4$$

$$r_z = 3.8" \quad r_y = 1.34"$$

$$e_z = 3.248"$$

$$J = 0.588 \text{ in}^4 \quad C_\omega = 2.764 \text{ in}^6$$

$$I_E = 117 \text{ in}^4 + 14.5 \text{ in}^4 + (8.1 \text{ in}^2)(3.248")^2 = 217 \text{ in}^4$$

$$\sigma_{xx_{cr}}^E = E \frac{\pi^2}{(L/r_y)^2} = 30,000 \frac{\pi^2}{L^2} (1.34")^2 = \frac{531,700}{L^2}$$

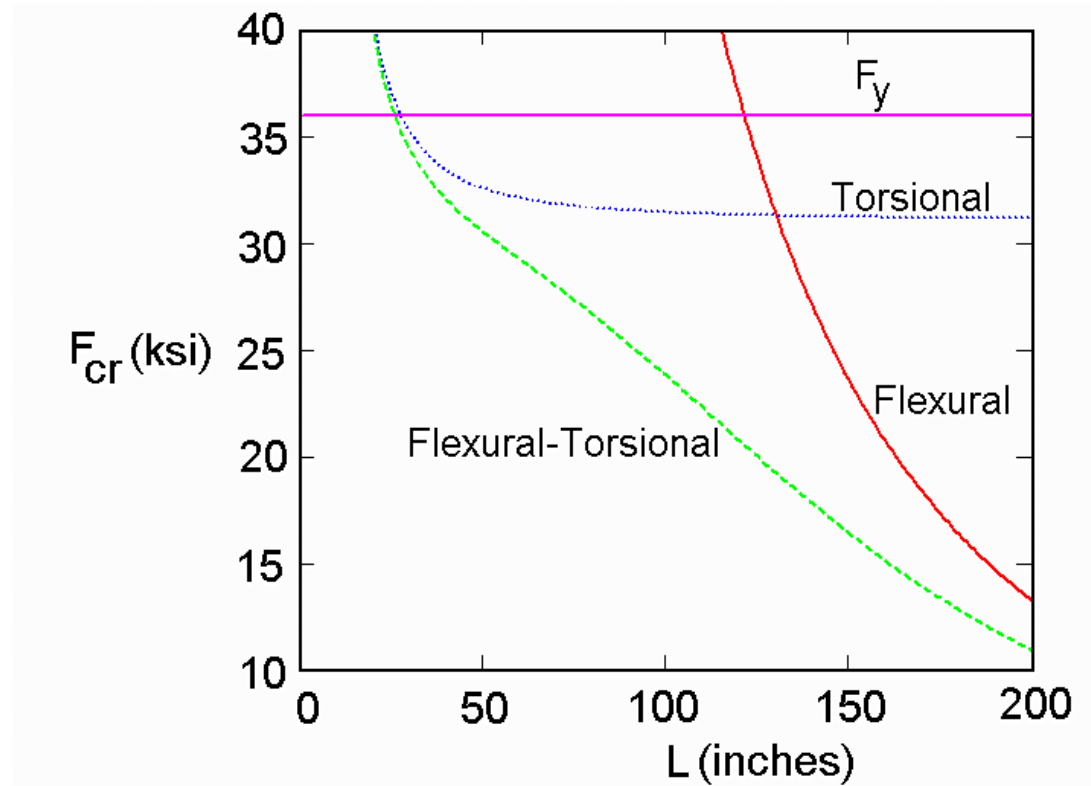
$$\begin{aligned} \sigma_{xx_{cr}}^{(T)} &= \frac{E}{I_E} \left( C_\omega \frac{\pi^2}{L^2} + \frac{J}{2(1+\nu)} \right) = \frac{30,000}{217} \left( 2.764 \frac{\pi^2}{L^2} + \frac{0.588}{2.6} \right) \\ &= \frac{3,771}{L^2} + 31.26 \end{aligned}$$

Substituting into the critical stress eqn yields

$$\sigma_{xx_{cr}}^{(FT)} = \frac{\left( \frac{535,500}{L^2} + 31.3 \right) - \sqrt{\left( \frac{535,500}{L^2} + 31.3 \right)^2 - \frac{1,289,300}{L^2} \left( \frac{3,771}{L^2} + 31.3 \right)}}{1.213}$$

This result is shown below in two plots.

- First plot shows the interaction of the critical stresses over a large theoretical range
- Second plot shows the interaction of the critical stresses over a practical range



Flexural-Torsional interaction significantly reduces the buckling load for this column, extending well into the elastic range!

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