9

Torsion of Closed Thin Wall (CTW) Sections

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§9.1. Introduction

This Lecture continues the treatment of Thin Wall (TW) sections in torsion. We proceed now to the case of Closed Thin Wall (CTW) sections. These are much more effective in resisting torsion because a *cell shear flow circuit* can be established to take up the internal torque.

§9.2. Closed TW Sections

A closed thin wall (TW) section is one in which *uninterrupted circuits for a shear flow q* can be established. This shear flow, which is an internal force per unit length tangential to the wall, resists the applied torque. Some examples of closed TW sections are depicted in Figure 9.1.

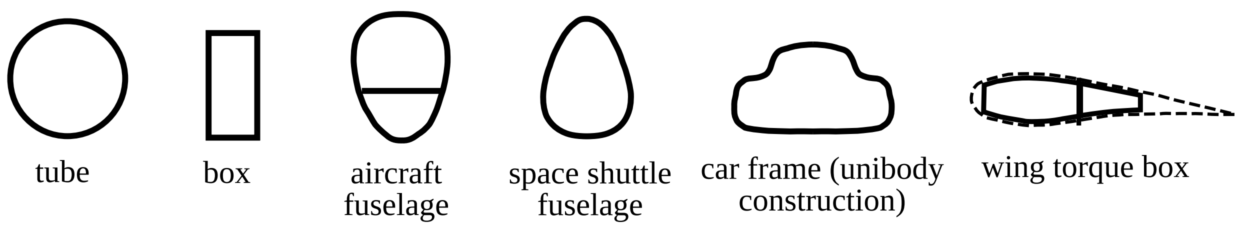


Figure 9.1. Closed thin wall section examples.

Such sections are *much more effective* in resisting torque than simularly configured open TW sections, sometimes by orders of magnitude.[1] This effectiveness is particularly important for aircraft wings because it determines the flutter and divergence speeds of the flight envelope.

If the wall thickness is much smaller than the other dimensions the shear flow q can be assumed to be *uniform* across the wall thickness t. The wall shear stress magnitude is then simply $\tau = q/t$. This approximation is sufficiently accurate for engineering purposes if the thickness does not exceed, say, 20% of the smallest cross section dimension. [2]

Closed TW sections are classified into *single cell* ot *multicellular*, depending on whether there is only one shear flow circuit or several. The necessary notation for the single cell case is introduced in Figure 9.2. Points along the wall midline are defined by s, called the arclength or "perimeter" coordinate. The wall thickness t may vary as the contour is traversed, i.e. $t = t(s)$, but the material is assumed to remain the same.

The stress analysis of the single cell case follows from elementary statics and is worked out on pages 189–191 of the Beer-Johnston-DeWolf textbook.[3] Here we only summarize the major steps. Let the distance from the midline to the center of application of the torque[4] be denoted by h, as shown in Figure 6(a). Then

$$T = \oint_s dT = \oint q(h\,ds) = q \oint h\,ds = q \oint 2\,dA_E = q(2A_E) = 2\,q\,A_E, \tag{9.1}$$

[1] Exercise 4.3 of HW#4 quantifies this statement for a wing torque box. This is also obvious in the relaative strength of the two cross sections tested in Experimental Lab 1.

[2] Discrepancies from the exact solution can be readily included in the safety factor against torsion failure.

[3] F. Beer, E. R. Johnston and J. T. DeWolf, *Mechanics of Materials*, McGraw-Hill, 4th ed., 2006

[4] This is called the *shear center* in a more advanced treatment of torsion.

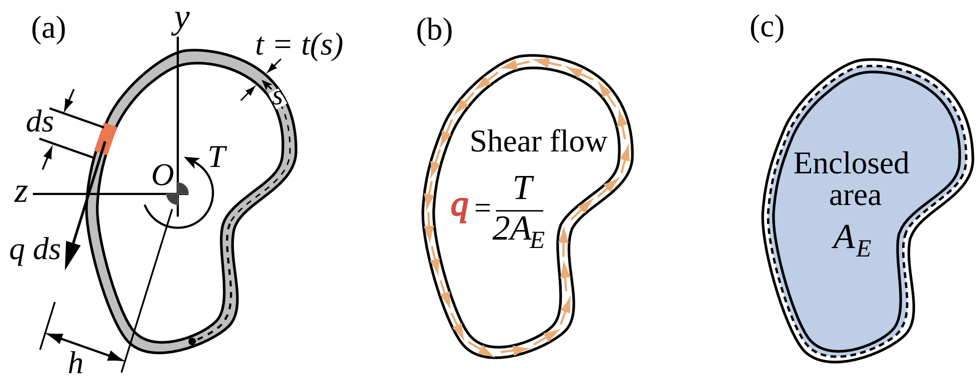


Figure 9.2. Torsion of single-cell closed TW section.

in which A_E is the *enclosed area* defined by the wall midline as illustrated in Figure 6(c). It is important not to confuse this with the cross section area, also called the *material area* or the *wall area*, which is that shaded in grey in Figure 9.2(a).

Solving (9.1) for q yields the shear flow and stress formulas

$$q = \frac{T}{2A_E}, \qquad \tau = \frac{q}{t} = \frac{T}{2t\,A_E} \tag{9.2}$$

Since q is constant along the contour, plainly the maximum shear stress occurs when the thickness is minimum:

$$\boxed{\tau_{max} = \frac{q}{t_{min}} = \frac{T}{2t_{min}A_E}} \tag{9.3}$$

It can be shown[5] that the twist-angle rate is given by

$$\boxed{\phi' = \frac{d\phi}{dx} = \frac{q}{2GA_E}\oint\frac{ds}{t} = \frac{T}{4GA_E^2}\oint\frac{ds}{t} = \frac{T}{GJ}, \qquad J = \frac{4A_E^2}{\oint\frac{ds}{t}}} \tag{9.4}$$

If the thickness t is uniform the contour integral in (9.4) reduces to p/t, where $p = \oint ds$ is the wall midline perimeter length, and the twist-rate expression simplifies to

$$\boxed{\phi' = \frac{d\phi}{dx} = \frac{q\,p}{2A_E\,t} = \frac{T}{GJ}, \qquad \text{in which} \qquad J = \frac{4A_E^2\,t}{p}.} \tag{9.5}$$

[5] Obtaining this relation requires energy methods, which is graduate level material.

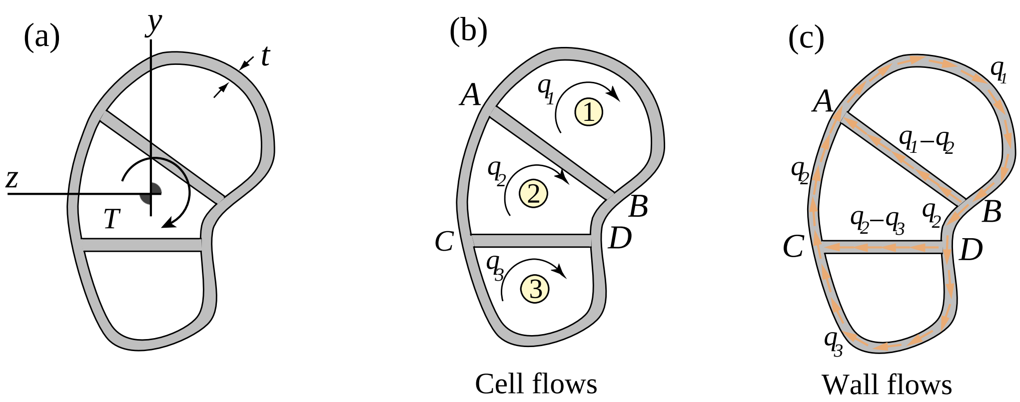


Figure 9.3. The torsion of multicellular closed TW sections is not covered in this course.

The analysis of a multicellular closed TW section under torsion, sketched in Figure 9.3, is far more elaborate because the problem is statically indeterminate, and is not considered in this course.

§9.3. Examples

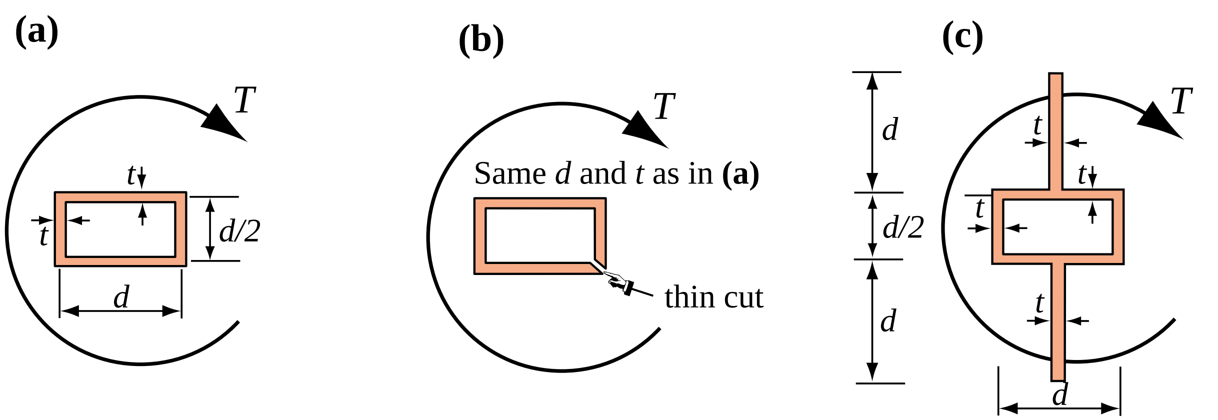


Figure 9.4. Shaft TW cross sections for examples in §9.3.

The folowing three problems were given in Midterm Exam 2, Fall 2005 (the third one was a Bonus Question).

§9.3.1. Closed Rectangular Tube

The cross section of this shaft is shown in Figure 9.4(a). Dimensions are $d = 2$ in and $t = 1/8$ in, d being a midline dimension. Same thickness t all around. The shear modulus is $G = 3.75 \times 10^6$ psi. The shaft is subjected to a torque of $T = 1600$ in-lb, which is uniform along a length of $L = 60$ in. Find (a) the maximum shear stress τ_{max} in psi (sign unimportant), and (b) the twist angle ϕ in both radians and degrees over the length L.

Solution. Using CTW theory:

(a) Enclosed area: $A_E = \frac{1}{2}d^2 = 2$ in^2. Shear flow: $q = T/(2A_E) = 1600/(2 \times 2) = 400$ lbs/in. Shear stress: $\tau = q/t = 400/(1/8) = 3200$ psi. This is also τ_{max} because the thickness t is uniform all around.

(b) The simplified formula (9.5) for J may be used because t is constant all around. Perimeter: $p = d + \frac{1}{2}d + d + \frac{1}{2}d = 3d = 6$ in. Torsional constant: $J = 4A_E^2\, t/p = 1/3$ in^4. Twist angle over length L: $\phi = T\,L/(G\,J) = 0.0768$ rad $= 4.40^\circ$.

§9.3.2. Slitted Rectangular Tube

The rectangular tube section of the previous example is cut along the length as sketched in Figure 9.4(b). The width of the cut is negligible. Required: (a) the maximum shear stress τ_{max} in psi (sign unimportant) (b) the twist angle ϕ in both radians and degrees over the length L; (c) compare the shear stress and twist angle to those found in the previous example.

Solution. Using OTW theory:

(a) Rectify into narrow rectangle with thickness $t = 1/8$ in and long dimension $b = p = d + \frac{1}{2}d + d + \frac{1}{2}d = 3d = 6$ in. Torsional constant: $J_\alpha \approx J_\beta \approx \frac{1}{3}\,(3d)\,t^3 = 1/256$ in^4. Max shear stress: $\tau_{max} = T t/J_\alpha = 51200$ psi.

(b) Twist angle over length L: $\phi = T\,L/(G\,J) = 6.991$ rad $\approx 400^\circ$.

(c) Maximum shear stress ratio: 51200/3200=16. Twist angle ratio: $400^\circ/4.40^\circ \approx 90$.

§9.3.3. A Hybrid Cross Section

Figure 9.4(c) shows a hybrid TW (HTW) section, in which two fins of the dimensions shown are attached to the rectangular tube of Figure 9.4(c). The material of the fins and tube is the same. Required: (a) the maximum shear stress τ_{max} in psi (sign unimportant) (b) the twist angle ϕ in both radians and degrees over the length L; (c) compare the shear stress and twist angle to those found in the CTW example.

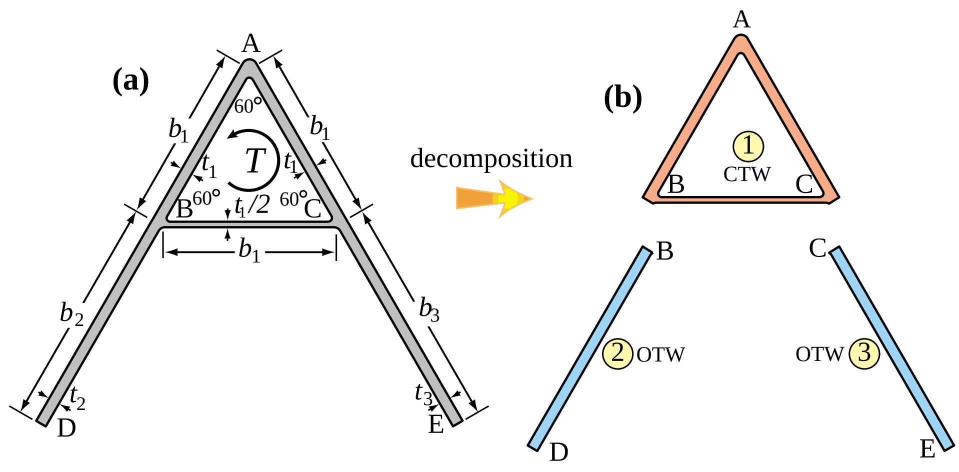
Solution. Use decomposition and apply CTW and OTW theory as appropriate. Start by decomposing the torque into the portions taken by the tube and the fins:

$$T = T_{tube} + T_{fin} + T_{fin} = T_{tube} + 2T_{fin}. \tag{9.6}$$

Next compute the torsional component for these components, using the approriate theory. For the tube, $J_{tube} = 4A_E^2\, t/p = 1/6$ in^4 from Example in §9.3.1. For a fin, $J_{fin} = \frac{1}{3}d\,t^3 = \frac{1}{3} \times 2 \times (1/8)^3 = 1/768$ in^4. The total J is $J_{tube} + 2J_{fin} = (65/384)$ in^4. Since G is the same for tube and fins, the torque portions are given by the J-ratios

$$T_{tube} = \frac{J_{tube}}{J} T = \frac{128}{130} \times 1600 = 1575.38 \text{ in-lb}, \quad T_{fin} = \frac{J_{fin}}{J} T = \frac{1}{130} \times 1600 = 12.31 \text{ in-lb} \tag{9.7}$$

(a) For the maximum shear stress, apply the CTW and OTW formulas for the tube and fin,



$$b_1 = 200\text{ mm},\quad b_2 = b_3 = 400\text{ mm},\quad t_1 = t_2 = t_3 = 6\text{ mm}$$
$$G = 80\text{ GPa (steel)},\quad T = 604\,10^3\text{ mm-N},\quad L = 2514\text{ mm}$$

Figure 9.5. A-shaped hybrid TW section under torsion for Example 9.2: (a) cross section, (b) decomposition into three portions.

respectively:

$$\begin{aligned} \tau_{max,tube} &= \frac{T_{tube}}{2A_E\,t} = \frac{1575.38}{2 \times 2 \times (1/8)} = 3151\text{ psi}, \\ \tau_{max,fin} &= \frac{T_{fin}\,t}{J_{fin}} = \frac{12.31 \times (1/8)}{1/768} = 1182\text{ psi}, \\ \tau_{max} &= \max(3151, 1182) = 3151\text{ psi} \quad \text{in tube.} \end{aligned} \tag{9.8}$$

(b) For the twist angle we use the total J computed above:

$$\phi = \frac{T\,L}{G\,J} = \frac{1600 \times 60}{3.75 \times 10^6 \times (65/384)} = 0.1511\text{ rad} = 8.66^\circ. \tag{9.9}$$

(c) Comparing the above results to those for the tube of Figure 9.4(a), it is obvious that the addition of the fins has not reduced the maximum shear and the twist angle by much (from 3200 psi to 3151 psi, and from 8.80° to 8.66°, respectively). This corroborates the well known fact that OTW components are relatively ineffective in resisting torsion.

§9.3.4. Recitation Problem Example

This is given as Problem # 3 of Recitation #4.

The "A-shaped" cross section of a HTW prismatic shaft is shown in Figure 9.5(a). It is subjected to uniform torque T over a distance L. The triangle ABC forms the CTW portion, whereas legs BD and CE collectively form the OTW portion. Data is given in metric units:

$$b_1 = 200\text{ mm},\quad b_2 = b_3 = 400\text{ mm},\quad t_1 = t_2 = t_3 = 6\text{ mm},$$
$$G = 80\text{ GPa (steel)},\quad T = 604\,10^3\text{ mm-N},\quad L = 2514\text{ mm},$$

in which the shear modulus G is the same for the whole section.

Find: (1) maximum overall shear stress τ_{max} in MPa, and (2) twist angle ϕ over length L in both radians and degrees.

Solution. Decompose into three portions as shown in Figure 9.5(b). Portion 1 is the CTW triangular box ABC. Portions 2 and 3 are the legs BD and CE, respectively. The total torque T is accordingly decomposed as

$$T = T_1 + T_2 + T_3$$

in which T_i denotes the torque taken by the i^{th} portion. To find these proceed as follows.

Compute the J_β of portion 1 using CTW theory:

$$A_E = \tfrac{1}{2} \times \text{base} \times \text{height} = \tfrac{1}{2} b_1 \,(b_1 \sin 60^\circ) = \tfrac{1}{4} b_1^2 \sqrt{3} = 17320 \text{ mm}^2,$$
$$J_{\beta 1} = \frac{4\, A_E^2}{\oint ds/t} = \frac{4 b_1^4 (3/16)}{b_1/t_1 + b_1/t_1 + b_1/(t_1/2)} = \frac{3}{16}\, b_1^3\, t_1 = 9\,10^6 \text{ mm}^4.$$

Here A_E denotes the enclosed area of the CTW portion. Next compute the J_β of portions 2 and 3 using OTW theory. Since the legs are identical:

$$J_{\beta 2} = J_{\beta 3} = \tfrac{1}{3} b_2\, t_2^3 = \tfrac{1}{3} b_3\, t_3^3 = 28800 \text{ mm}^4.$$

Here $\beta \approx \frac{1}{3}$ is used for simplicity since the b/t ratios of the OTW rectangles exceed 10. The total J_β for the whole section is

$$J_\beta = J_{\beta 1} + J_{\beta 2} + J_{\beta 3} = 9.0576\,10^6 \text{ mm}^4$$

Because G is the same for all portions, the ratios $J_{\beta i}/J_\beta$, $i = 1, 2, 3$, determine how much torque is taken up by each portion:

$$T_1 = \frac{9\,10^6}{9.0576\,10^6} T = 600159 \text{ mm-N}, \qquad T_2 = T_3 = \frac{28800}{9.0576\,10^6} T = 1920.51 \text{ mm-N}$$

For stress calculations we may take $J_{\alpha 2} = J_{\beta 2}$ and $J_{\alpha 3} = J_{\beta 3}$ since $\alpha \approx \beta \approx \frac{1}{3}$ when the b/t of an OTW rectangle exceeds 10. The maximum shear stresses over each portion are

$$\tau_{max1} = \frac{T_1}{2 A_E\, (t_1/2)} = 5.775 \text{ MPa}, \qquad \tau_{max2} = \tau_{max3} = \frac{T_2 t_2}{J_{\alpha 2}} = \frac{T_3 t_3}{J_{\alpha 3}} = 0.4001 \text{ MPa}$$

whence the overall maximum shear is

$$\boxed{\tau_{max} = \max\,(\tau_{max1}, \tau_{max2}, \tau_{max3}) = 5.775 \text{ MPa}}$$

This occurs over segment BC of the CTW portion because that segment has the minimum thickness ($t_1/2 = 3$ mm) when traversing that circuit.

The twist rate of the whole section is

$$\frac{d\phi}{dx} = \frac{T}{G\, J_\beta} = 8.33554\,10^{-7} \text{ rad/mm}$$

Since G, T and J_β are constant over the shaft length L, the end-to-end twist angle is

$$\boxed{\phi = \frac{d\phi}{dx} L = \frac{T\, L}{G\, J_\beta} = 0.00209556 \text{ rad} = 0.120066^\circ}$$