8

Torsion of Open Thin Wall (OTW) Sections

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§8.1. Introduction

The treatment of torsion in most Mechanics of Materials textbooks is uneven. While circular cross sections are analyzed in detail (sometimes too thoroughly), the coverage of non-circular sections is typically relegated to “advanced material” covering 4-5 pages. (In this respect Beer-Johnston-DeWolf is somewhat better than Vable, who totally omits Open Thin Wall sections)

Lectures 8–9 try to fill those gaps by collecting basic material on torsion of thin-wall (TW) sections as needed for this course. They cover open Thin Wall (OTW) and Closed Thin Wall (CTW) sections, respectively. These Lectures also provide the background theory required for the torsion experiment (Experimental Lab 1).

The accurate determination of stresses, strains and displacements for torqued members of *non-circular* cross sections in the elastic range is analytically involved because it requires the solution of a *partial differential equation* (either St-Venant’s or Prandtl’s PDE) supplied by the Theory of Elasticity. Such advanced material is covered in graduate courses on structures.

The analysis can be simplified considerably, however, in the case of *thin wall* (TW) sections. These are members with cross section consisting, well, of thin walls. To qualify as “thin” the wall thicknesses should be small, typically 10% or less, when compared to the overall cross section dimensions.

Structural members with thin wall cross sections are common in aerospace, in both aircraft and space systems, because of two obvious advantages:

- Weight savings
- Providing room, as well as protective coverage, for non-structural parts such as fuel conduits, avionics equipment, power cables, etc

Thin-wall structural members are often available as commercial profiles fabricated with steel, aluminum or titanium alloys, or (at least in part) with laminated composites.

§8.2. TW Section Classification

From the standpoint of torsion resistance, thin wall (TW) sections are classified into three types:

- *Closed Thin Wall* or CTW: at least one *cell shear-flow circuit* can be established in the cross section.
- *Open Thin Wall* or OTW: a cell shear-flow circuit cannot be established.*
- *Hybrid Thin Wall* or HTW: the section contains a mixture of CTW and OTW components.

Closed TW sections are in turn classified into *single cell* or *multicell*, according to whether one or several shear flow circuits, respectively, can be identified.

The examples and figures that follow should make the distinction clear. In this lecture we deal only with *open* TW sections. Examples of such sections are depicted in Figure 8.1 on next page.

Complicated OTW sections can often be *decomposed into rectangles* as described below in this Lecture, using “rectification” of curved walls of constant thickness if necessary. Accordingly we treat the case of torsion of a solid rectangular cross section first.

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* Even in OTW sections, a shear flow has to form because of the internal reaction to the applied torque, but it does not flow around a cell.

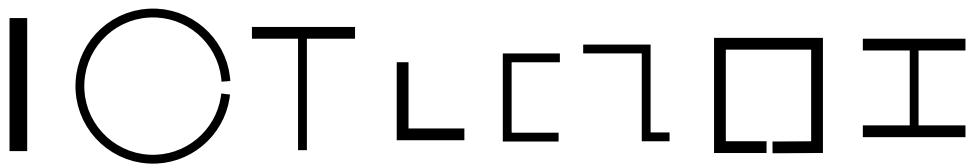

FIGURE 8.1. Open thin wall section examples.

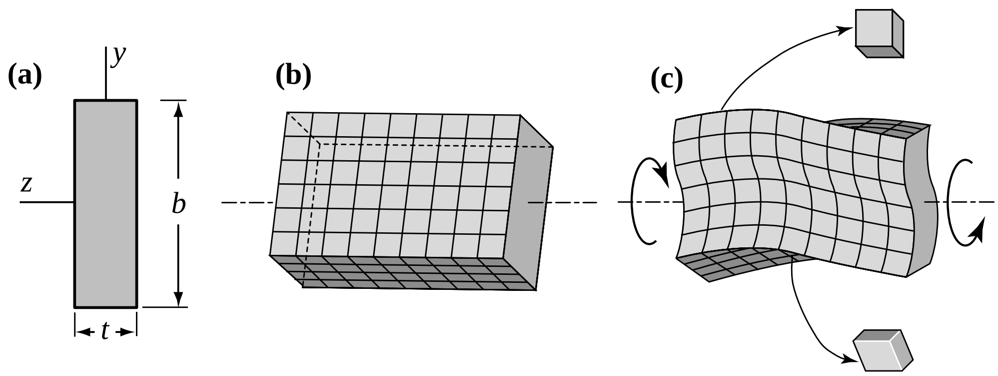

FIGURE 8.2. Rectangular shaft: (a) cross section dimensions; (b) before and (c) after a torque is applied.

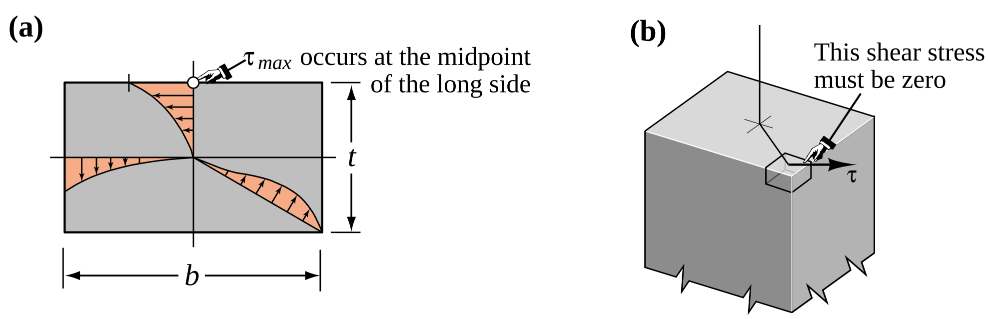

FIGURE 8.3. Non-thin rectangular shaft under torque: (a) shear stress distribution along three radial lines, (b) why shear stress at a corner must be zero.

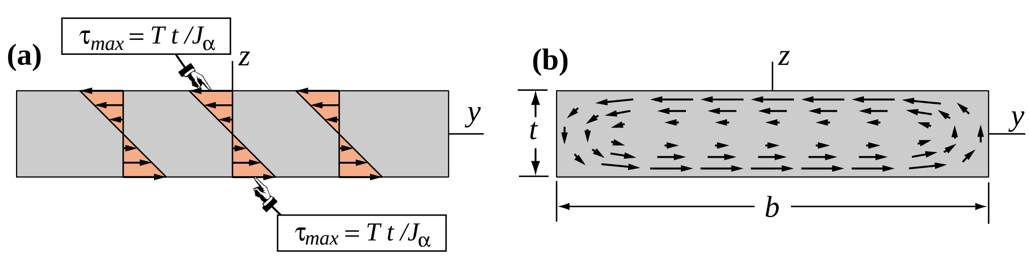

FIGURE 8.4. Thin rectangular shaft under torque: (a) shear stress distribution along most of long side, (b) actual shear stress distribution showing end effects in the vicinity of the short sides.

§8.3. Open TW Sections: Rectangle

This section covers the rectangular cross section in some detail. This provides a "divide and conquer" tool for the treatment of more general open TW sections, since (as noted) those can be often decomposed into rectangles.

§8.3.1. Behavior of Torqued Member with Solid Rectangular Cross Section

Consider a shaft with a *rectangular* cross section such as that shown in Figure 8.2(a). The long dimension b is oriented along y. The short dimension t is called the *thickness*, which is oriented along z. (The section pictured here is not that of a thin rectangle, because behavior visualization is easier if b and t are of comparable magnitude.)

The untorqued shaft is pictured in Figure 8.2(b). Upon applying a torque, it deforms as sketched in Figure 8.2(c). Figure 8.3(a) shows the shear stress distribution along three radial lines emanating from the center. Two important differences with the case of circular cross section shaft studied in Lecture 7 are:

- Cross sections do not remain plane but *warp*. The nature of the distortions can be surmised from Figure 8.2(c). This is actually the case for any non-circular cross section.
- In a shaft of circular cross section, the maximum shear stress occurs at all points located at the greatest distance from the center. In a rectangle the maximum occurs at the midpoint of the long sides, as indicated in Figure 8.3(a). At the 4 rectangle corners, which are the points located at the greatest distance from the center, shear stresses vanish. The reason is obvious on studying Figure 8.3(b): if the shown τ is nonzero there would be a reciprocal shear stress acting along the surface of the shaft, which is impossible since that surface is load-free.

The rectangle is called *narrow* or *thin* if t is much smaller than b in the sense that

$$t \le b/10 \quad \text{or} \quad b/t \ge 10 \tag{8.1}$$

If (8.1) holds, the only important shear stress component in the x, y, z system is τ_{xy}. This stress varies approximately as a linear function of z. That means that it is zero at $z = 0$ (the rectangle long midline) and is maximum on both surfaces $z = \pm t/2$. It is also approximately constant along the long rectangle dimension except in the vicinity of the edges $y = \pm b/2$, where it drops. See Figure 8.4(b).

§8.3.2. Stress and Twist-Rate Formulas for Rectangle

If the rectangle is not narrow, the solution to the torsion problem has to be obtained using the Theory of Elasticity.† Let as usual T denote the applied torque, $\phi = \phi(x)$ the twist angle and G the shear modulus. The maximum shear stress and the twist rate (twist-angle-per-unit-length) formulas furnished by the exact theory can be summarized as follows.

$$\boxed{\tau_{max} = \frac{Tt}{J_\alpha}, \qquad \phi' = \frac{d\phi}{dx} = \frac{T}{G J_\beta}, \qquad \text{in which } J_\alpha = \alpha b t^3 \text{ and } J_\beta = \beta b t^3.} \tag{8.2}$$

† See, e.g., S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, 3rd ed., McGraw-Hill, New York, 1969, Sec 109.

in which a prime denotes derivative with respect to x. Here α and β are numerical coefficients that depend on the ratio b/t, as given in the following table[‡]

b/t	1.0	1.2	1.5	2.0	2.5	3.0	4.0	5.0	6.0	10.0	∞
α	0.208	0.219	0.231	0.246	0.258	0.267	0.282	0.291	0.299	0.312	1/3
β	0.141	0.166	0.196	0.229	0.249	0.263	0.281	0.291	0.299	0.312	1/3

(8.3)

For $b/t \geq 3$, the two coefficients agree to at least 2 places, and may be linearly interpolated in t/b as

$$\alpha = \beta \approx \frac{1}{3} - 0.210\,\frac{t}{b}, \tag{8.4}$$

More accurate interpolations given in Roark's manual[†] are

$$\begin{aligned} \alpha &= \frac{1}{3}\,\frac{1}{1 + 0.6095\frac{t}{b} + 0.8865\left(\frac{t}{b}\right)^2 - 1.8025\left(\frac{t}{b}\right)^3 + 0.9100\left(\frac{t}{b}\right)^4}, \\ \beta &= \frac{1}{3} - 0.210\frac{t}{b}\left[1 - \frac{1}{12}\left(\frac{t}{b}\right)^4\right], \end{aligned} \tag{8.5}$$

These fit the data in (8.3) to at least 3 places for all aspect ratios $b/t \geq 1$.

If the rectangle qualifies as narrow in the sense (8.1), one can adopt the easier to remember but rougher approximation

$$\boxed{\alpha = \beta \approx \frac{1}{3}, \quad \text{whence} \quad J = J_\alpha = J_\beta \approx \tfrac{1}{3} b t^3,} \tag{8.6}$$

This has error $< 7\%$ if $b/t > 10$. Such discrepancy is easily covered with safety factors. In particular, the approximation (8.6) is sufficient for the theory portion of Lab 1.

The torsional constant J that appears in (8.3) and (8.6) is *not* the polar moment of inertia $\int_A \rho^2\,dA$ as was the case for the circular cross sections covered in Lecture 7.* But it has the same physical dimensions: length to the fourth power. As in the case of circular sections, the product GJ that appears in the expression of $\phi' = d\phi/dx = T/(GJ)$ is called the *torsional rigidity*. We shall use the symbol J consistently for that purpose.

As previouly noted, the maximum shear occurs at the boundary points ($y = 0$, $z = \pm t/2$) closest to the rectangle center. For narrow rectangles that value is approximately the same over the long side surfaces $z = \pm t/2$ except in the vicinity of the short sides; cf. Figure 8.4.

‡ From E. P. Popov, *Engineering Mechanics of Solids*, Prentice Hall, 1999, p. 243.

† W. C. Young, *Roark's Formulas for Stress & Strain*, McGraw-Hill, 6^{th} ed., 1989, p. 348.

* It can be shown that J is always less than the polar moment of inertia except for circular cross sections. The proof of this property, however, require mathematical knowledge beyond that assumed for this course.

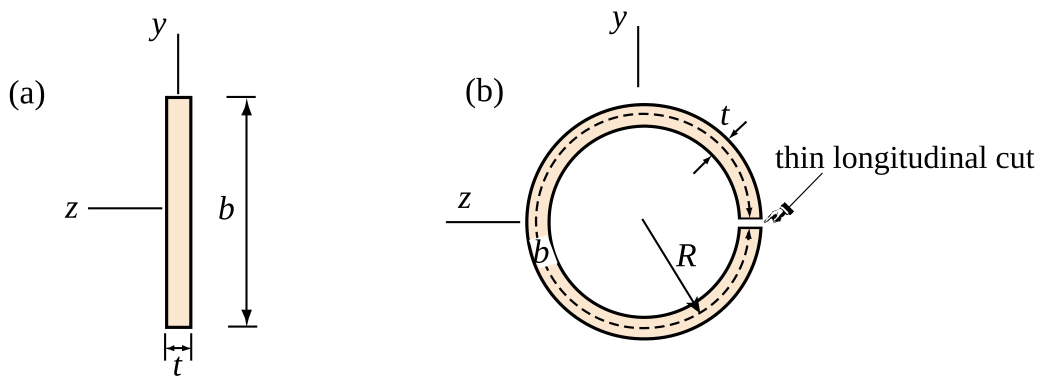

FIGURE 8.5. Narrow rectangle and slotted tube (cut-along) tube sections can be treated by the same method.

§8.4. Rectifiable OTW Sections

The solution (8.2)–(8.6) applies to any open TW profile than can be "rectified" into a *narrow* rectangle. For example, the tube shown in Figure 8.5(b) has a thin longitudinal cut that makes it an open TW section. That solution can be used applies if one takes $b = 2\pi R$, where R is the mid-radius; this of course assumes that the cut is of negligible width and that $b/t \geq 10$.

If the rectangle is not sufficiently narrow the equivalence should be used with caution, and abandoned all together is b/t is less than, say, 2. But many shaft profiles do satisfy that geometric restriction.

This method can be also used for channel (C), L or Z profiles, because those can be rectified into rectangles. This equivalence holds as long as the profile is fabricated of the same material throughout, and has a *uniform wall thickness*. Else the decomposition-into-rectangles method outlined below for the T section should be used.

The example of a Z profile with varying wall thickness is treated in §8.6. Because the thickness is not constant, it cannot be rectified into just one rectangle.

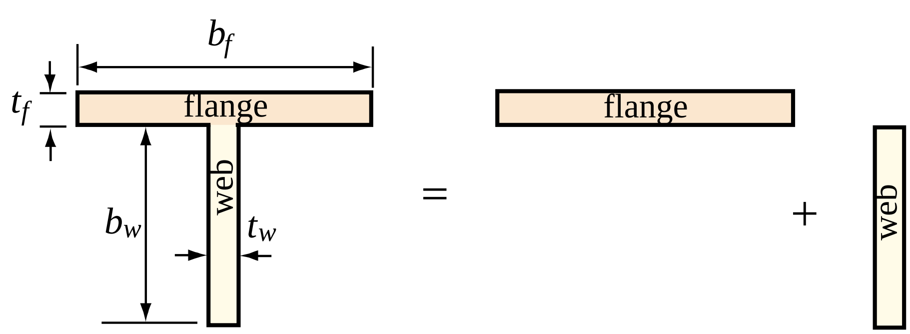

FIGURE 8.6. Decomposition of a thin-wall T profile into two rectangles for torsion analysis.

§8.5. Torsion of a T Profile

Open TW profiles that cannot be rectified into a single narrow rectangle must be decomposed as the sum of such. This is the case for T and double-T sections. The procedure is illustrated for the T cross section pictured in Figure 8.6.

The profile is decomposed into two narrow rectangles: flange and web, with dimensions shown in Figure 8.6. We shall assume that the material of the web and flange is the same, with shear modulus G. Decompose the total torque T into two parts: T_f and T_w taken by the flange and web, respectively, and express that the twist rate of each component is the same:

$$\boxed{T = T_f + T_w, \qquad \frac{d\phi}{dx} = \frac{d\phi_f}{dx} = \frac{T_f}{G J_{\beta f}} = \frac{d\phi_w}{dx} = \frac{T_w}{G J_{\beta w}}.} \tag{8.7}$$

Here $J_{\beta f} = \beta_f b_f t_f^3$ and $J_{\beta w} = \beta_w b_w t_w^3$, in which coefficients β_f and β_w are obtained from Table (8.5) as function of the aspect ratios b_f/t_f and b_w/t_w, respectively.

The second relation in (8.7) assumes that the twist angle is the same for flange and web: $\phi_f = \phi_w = \phi$. This relation yields $T_f = G J_{\beta f}\, d\phi/dx$ and $T_w = G J_{\beta w}\, d\phi/dx$. Adding gives $T = T_f + T_w = G(J_{\beta f} + J_{\beta w})\, d\phi/dx = G J_\beta\, d\phi/dx$. Consequently, to compute

$$\boxed{J_\beta = J_{\beta w} + J_{\beta f}} \tag{8.8}$$

for the T section we simply *add the contributions of the flange and web*. The rigidity of the complete T section is $G J_\beta$ and the twist angle rate is given by $d\phi/dx = T/(G J_\beta)$.

The distribution of shear stresses noted previously for the thin rectangle *applies to each rectangular component in turn* if sufficiently away from "end effects.* First, using the foregoing results, compute the torques taken up by the flange and web, T_f and T_w:

$$T_f = \frac{J_{\beta f}}{J_\beta} T = \frac{J_{\beta f}}{J_{\beta f} + J_{\beta w}} T, \qquad T_w = \frac{J_{\beta w}}{J_\beta} T = \frac{J_{\beta w}}{J_{\beta f} + J_{\beta w}} T. \tag{8.9}$$

Then the maximum shear stresses stresses on the surface of the web and flange can be written

$$\boxed{\tau_{max\ f} = \frac{T_f t_f}{J_{\alpha f}}, \quad \tau_{max\ w} = \frac{T_w t_w}{J_{\alpha w}}, \quad \text{whence} \quad \tau_{max} = \max(\tau_{max\ f}, \tau_{max\ w}).} \tag{8.10}$$

Here $J_{\alpha f} = \alpha_f b_f t_f^3$ and $J_{\alpha w} = \alpha_w b_w t_w^3$, in which coefficients α_f and α_w are obtained from the table (8.4) as functions of the aspect ratios b_f/t_f and b_w/t_w, respectively.

If both rectangles are sufficiently narrow (as in the case of the OTW section tested in Lab 1), one can take $\alpha_f = \alpha_w = \beta_f = \beta_w \approx \frac{1}{3}$, in which case $J_f = J_{\alpha f} = J_{\beta f} = \frac{1}{3} b_f t_f^3$, and $J_w = J_{\alpha w} = J_{\beta w} = \frac{1}{3} b_w t_w^3$, and the foregoing equations can be considerably simplified. These simplifications are studied in further generality in §8.8.

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* For the T section once such end effect is the junction between web and flange. At the juncture corners stress concentration may appear if they are not appropriately rounded with fillets.

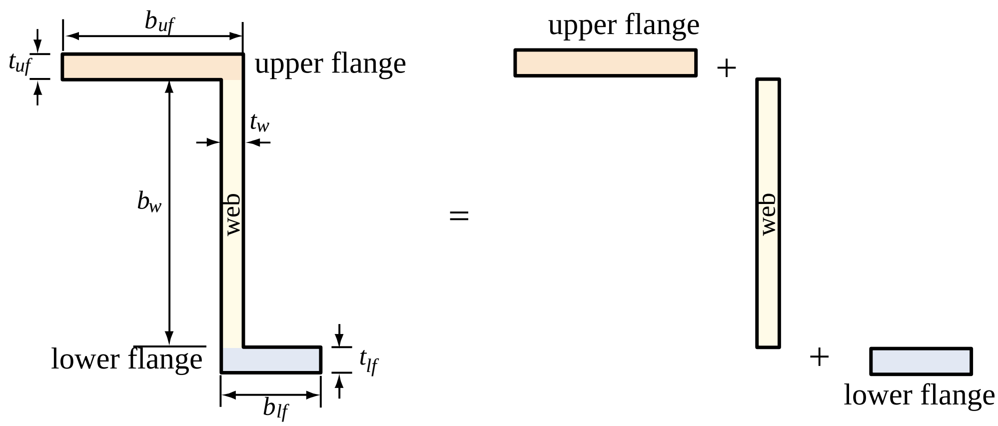

FIGURE 8.7. Decomposition of a thin-wall T profile into two rectangles for torsion analysis.

§8.6. A Z Profile with Variable Wall Thickness

It was noted above that OTW sections such as Z and C (channel) profiles can be rectified into a single narrow rectangle, as long as the material and wall thickness is the same throughout. In most commercial profiles the material is the same, so that is not an important restriction. If the wall thickness varies, however, the decomposition-into-rectangles method should be used.

As an example, Figure 8.7 depicts a Z profile whose upper flange, lower flange and web have different thicknesses: t_{uw}, t_{lw} and t_w, respectively. The section is decomposed into three narrow rectangles as shown there. Consequently

$$J_\beta = J_{\beta uf} + J_{\beta lf} + J_{\beta w}, \quad J_{\beta uf} = \beta_{uw} b_{uf} t_{uf}^3, \quad J_{\beta lf} = \beta_{lw} b_{lf} t_{lf}^3, \quad J_{\beta w} = \beta_w b_w t_w^3, \tag{8.11}$$

where the βs are obtained from the table (8.3) according to aspect ratios b_{uf}/t_{uf}, etc. If all rectangles are sufficient narrow, take simply $\beta_{uf} = \beta_{lf} = \beta_w = \frac{1}{3}$. Using the values (8.11) one calculates the torques T_{uf}, T_{lf} and T_w taken up by the upper flange, lower flange and web, respectively, as $T_{uf} = (J_{\beta uf}/J_\beta)T$, $T_{lf} = (J_{\beta lf}/J_\beta)T$ and $T_w = (J_{\beta w}/J_\beta)T$. As a check, $T_{uf} + T_{lf} + T_w = T$. The calculation of maximum shear stress follows the same procedure previously outlined for the T section.

§8.7. OTW Torsion Examples

For now, one example. More to be eventually added.

§8.7.1. A Double T Steel Beam

Figure 8.8 shows the dimensions (in inches) of a W12 × 65 rolled-steel beam. Using the approximate values $\alpha = \beta = \frac{1}{3}$ for the coefficients in (8.2), compute

(a) The torsional constant $J = J_\beta = J_\alpha$, and compare with the value of $J = 2.18$ in^4 given in the *AISC Manual of Steel Construction*.

(b) The torsional rigidity GJ if $G = 12 \times 10^6$ psi.

(c) The maximum torque T_{max} in lb-in that the beam can take if the yield stress in shear is $\tau_Y = 36$ ksi and the safety factor against torque yield is $s_F = 4$.

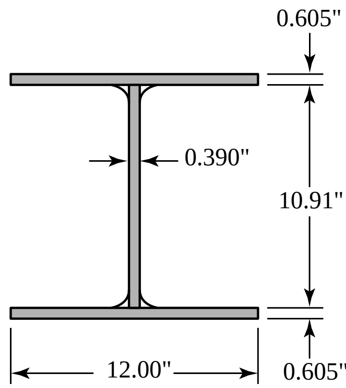

FIGURE 8.8. A W12× 65 rolled-steel beam profile for example in §8.7.1.

Solution

(a) The cross section can be divided into three rectangles: two flanges and one web. The flanges are identical. Using the approximate formula $J \approx \frac{1}{3}bt^3$ for each rectangle we get

$$J = \frac{1}{3}\left[(2 \times 12 \times 0.605^3) + 10.91 \times 0.390^3\right] = 1.99 \text{ in}^4 \tag{8.12}$$

The value given in the *AISC Manual* is larger: 2.18 in^4. The discrepancy may be attributed to neglecting the fillets at the four inside corners; see Figure 8.8.

(b) The torsional rigidity is $GJ = 12 \times 10^6$ psi $\times$ 1.99 $\text{in}^4 \approx 24 \times 10^6$ in^2-lb.

(c) If the applied torque is T (in-lb) the maximum shear stress will occur in the rectangle of maximum thickness (this result is proven in §8.8), that is, one of the flanges. The maximum shear in either flange is

$$\tau_{max} = \frac{T_f\, t_f}{J_f} = \frac{T\, t_f}{J} = \frac{T \text{ (lb-in)} \times 0.605 \text{ in}}{1.99 \text{ in}^4} = 0.304\, T \text{ psi} . \tag{8.13}$$

Applying the given safety factor gives

$$\tau_{max} = 0.304\, T \le \frac{\tau_Y}{s_F} = 9 \text{ ksi} = 9{,}000 \text{ psi} \quad \Rightarrow \quad T_{max} = \frac{9{,}000}{0.304} = 29{,}605 \text{ lb-in.} \tag{8.14}$$

§8.8. General Decomposition of an OTW Section

(This is backup material, not covered in class. Included here because no textbook has it.).

Suppose a member with OTW section subject to torque T has been decomposed into n rectangles. The properties of the i^{th} rectangle are denoted as follows: b_i and t_i are the long dimension and thickness, respectively, with $b_i \ge t_i$; G_i is the shear modulus; $J_{\alpha,i}$ and $J_{\beta,i}$ are the inertia-like constants defined in (8.2); α_i and β_i are dimensionless functions of the aspect ratio b_i/t_i tabulated in (8.3); and T_i is the portion of total torque T taken by that rectangle.

The twist rate $d\phi/dx = \phi'$ is assumed to be the *same for all rectangles* and linked to the portion of torque taken by each through the stiffness relation

$$\phi' = \frac{T_i}{G_i\, J_{\beta,i}}, \quad i = 1, \ldots n, \tag{8.15}$$

Assumption (8.15) provides $n-1$ independent equations, which in addition to the equilibrium condition

$$T = \sum_{i=1}^{n} T_i \tag{8.16}$$

gives n equations for the n unknowns T_i. The quickest way to solve this system is as follows. Express $T_i = G_i\, J_{\beta,i}\, \phi'$ from (8.15), replace into (8.16), factor out the twist rate and solve for it:

$$T = \left(\sum_{i=1}^{n} G_i\, J_{\beta,i} \right) \phi' \stackrel{\text{def}}{=} (G J_\beta)\, \phi' \quad \Rightarrow \quad \phi' = \frac{T}{G J_\beta} \tag{8.17}$$

in which $G J_\beta = \sum_{i=1}^{n} G_i\, J_{\beta,i}$ is the total torsional rigidity of the member. Consequently the torque portions are given explicitly as the rigidity ratios

$$T_i = \frac{G_i\, J_{\beta,i}}{G J_\beta}\, T, \quad i = 1, \ldots n. \tag{8.18}$$

(Note that it is impossible to separate G from $G J_\beta$ unless the shear modulus is the same for all rectangles, a common case considered below.) The maximum shear stress in the i^{th} rectangle is $\tau_{max,i} = T_i\, t_i / J_{\alpha,i}$ and the overall maximum is

$$\tau_{max} = \max_i \left(\tau_{max,i}\right) = \max_i \left(\frac{T_i\, t_i}{J_{\alpha,i}} \right), \quad i = 1, \ldots n. \tag{8.19}$$

Several simplifications are worth noting. First, if all rectangles are made of the same material so that $G_i = G$ for all i, the shear modulus cancels out in (8.18), which reduces to

$$T_i = \frac{J_{\beta,i}}{J_\beta}\, T, \quad i = 1, \ldots n, \quad \text{in which} \quad J_\beta = \sum_{i=1}^{n} J_{\beta,i} \tag{8.20}$$

Next, if all rectangles are so narrow that we can take $\alpha = \beta = \frac{1}{3}$, then $J_{\beta,i} = J_{\alpha,i} = J_i = \frac{1}{3} b_i\, t_i^3$, whence α and β may be dropped from the J subscripts. The total J is simply $J = \sum_{i=1}^{n} J_i$ and

$$T_i = \frac{J_i}{J}\, T, \quad \tau_{max} = \max_i \left(\frac{T_i\, t_i}{J_i} \right) = \max_i \left(\frac{T\, t_i}{J} \right) = \frac{T}{J} \max_i\, t_i, \quad i = 1, \ldots n. \tag{8.21}$$

The last form of τ_{max} says that the *largest shear stress will occur wherever the thickness is larger*.

Finally, if all rectangles have the common thickness t,

$$J = \sum_{i=1}^{n} \tfrac{1}{3} b_i t^3 = \tfrac{1}{3} b\, t^3, \quad \text{in which} \quad b = \sum_{i=1}^{n} b_i, \tag{8.22}$$

which is the "rectification recipe:" the n rectangles may be coalesced into one with long dimension b equal to the sum of all b_i. Observe that for this to be valid three conditions must hold: (a) the shear modulus must be the same throughout, (b) all rectangles must have the same thickness, and (c) all rectangles must be so thin than $\alpha = \beta \approx \frac{1}{3}$.